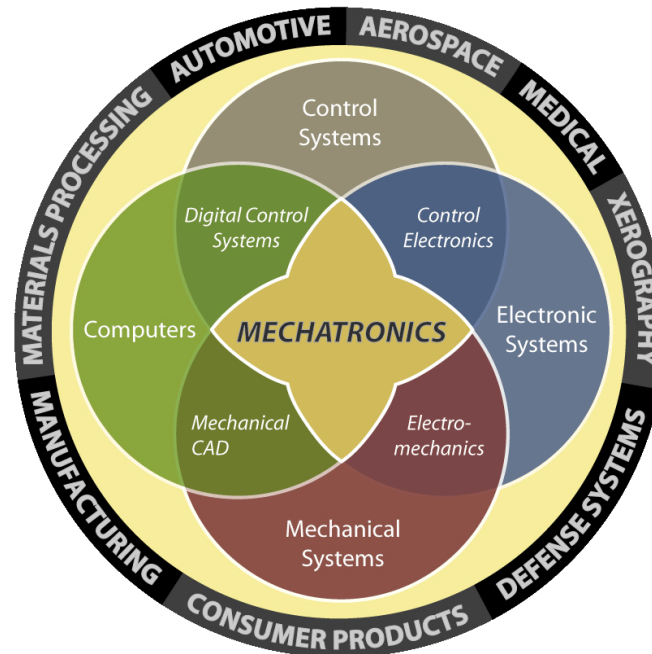


# Performance Specifications



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# Performance Specifications of Closed-Loop Systems

- Basic Considerations
- Time-Domain Performance Specifications
- Frequency-Domain Performance Specifications

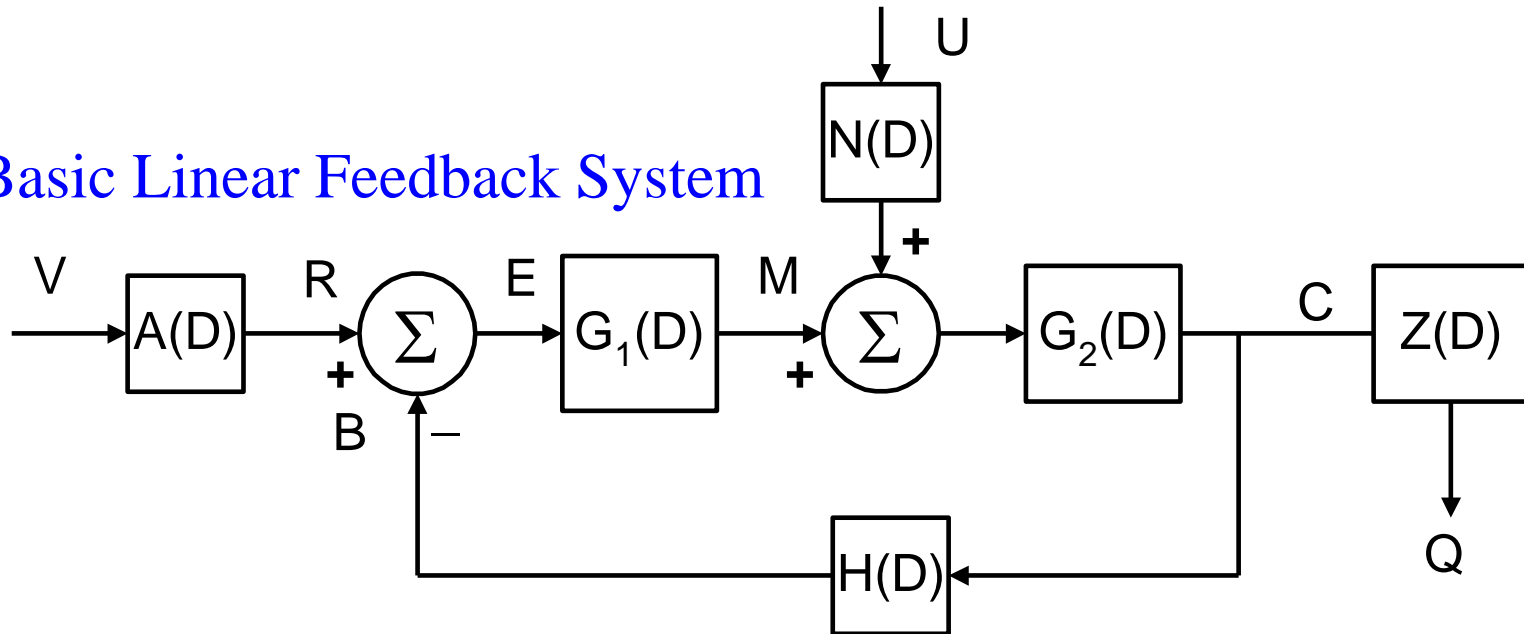
# Basic Considerations

- Most of our discussion will involve rather specific mathematical performance criteria whereas the ultimate success of a controlled process generally rests on economic considerations which are difficult to calculate.
- This rather nebulous connection between the technical criteria used for system design and the overall economic performance of the manufacturing unit results in the need for much exercise of judgment and experience in decision making at the higher management levels.

- Control system designers must be cognizant of these higher-level considerations but they usually employ **rather specific** and **relatively simple** performance criteria when evaluating their designs.

- Control System Objective
  - C follow desired value  $V$  and ignore disturbance  $U$
  - Technical performance criteria must have to do with how well these two objectives are attained
- Performance depends both on system characteristics and the nature of  $V$  and  $U$ .

### Basic Linear Feedback System



- The practical difficulty is that precise mathematical functions for  $V$  and  $U$  will not generally be known in practice.
- Therefore the random nature of many practical commands and disturbances makes difficult the development of performance criteria based on the actual  $V$  and  $U$  experienced by real system.
- It is thus much more common to base performance evaluation on system response to simple "standard" inputs such as steps, ramps, and sine waves.

- This approach has been successful for several reasons:
  - In many areas, experience with the actual performance of various classes of control systems has established a good correlation between the response of systems to standard inputs and the capability of the systems to accomplish their required tasks.
  - Design is much concerned with comparison of competitive systems. This comparison can often be made nearly as well in terms of standard inputs as for real inputs.

- Simplicity of form of standard inputs facilitates mathematical analysis and experimental verifications.
- For linear systems with constant coefficients, theory shows that the response to a standard input of frequency content adequate to exercise all significant system dynamics can then be used to find mathematically the response to any form of input.



- Standard performance criteria may be classified as falling into two categories:
  - **Time-Domain Specifications:** Response to steps, ramps, and the like
  - **Frequency-Domain Specifications:** Concerned with certain characteristics of the system frequency response
- Both time-domain and frequency-domain design criteria generally are intended to specify one or the other of:
  - speed of response
  - relative stability
  - steady-state errors

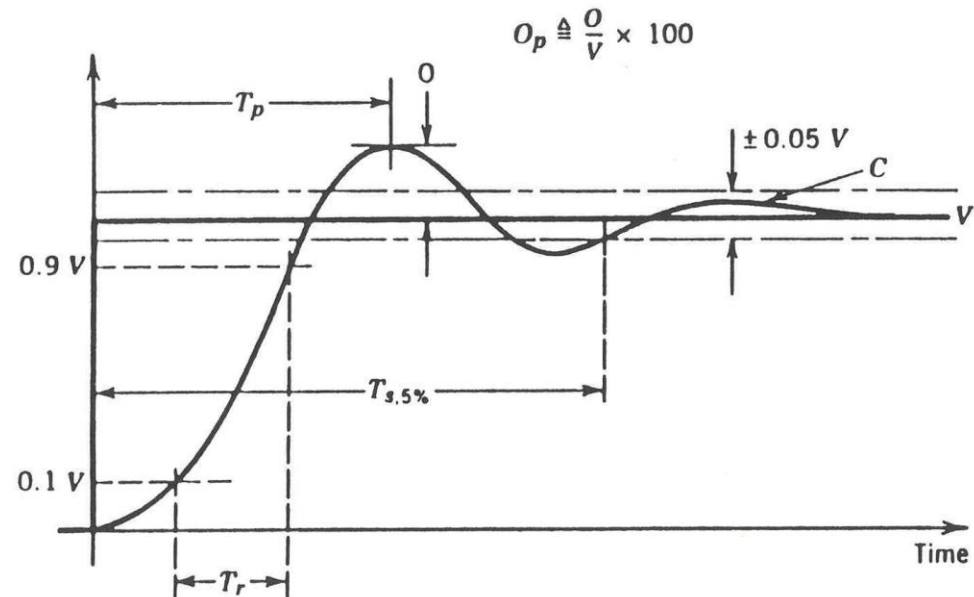
- Both types of specifications are often applied to the same system to ensure that certain behavior characteristics will be obtained.
- All performance specifications are meaningless unless the system is absolutely stable. So we assume absolute stability for the remainder of this discussion.

# Time-Domain Performance Specifications

- For linear systems, the superposition principle allows us to consider response to commands apart from response to disturbances.
- If both occur simultaneously, the total response is just the superposition of the two individual responses.
- In nonlinear systems, such treatment with subsequent superposition is not valid.

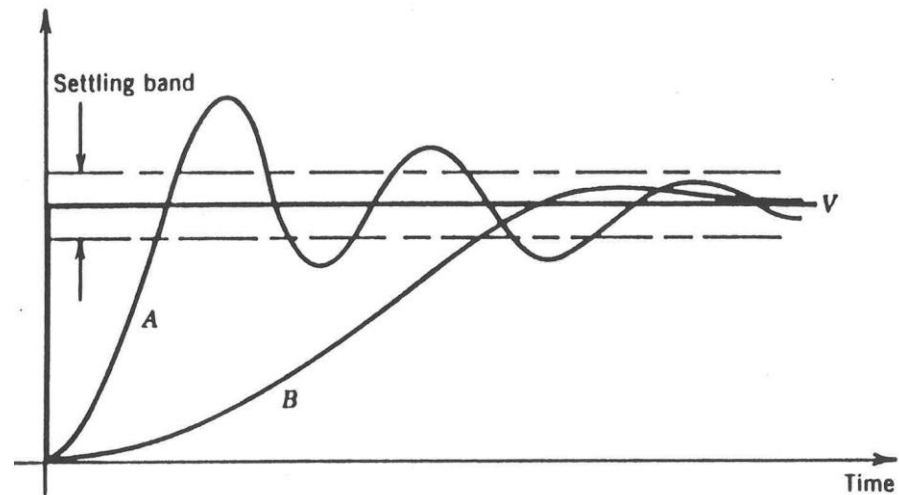
- Rise time,  $T_r$ , and peak time,  $T_p$ , are speed of response criteria.
- Percent overshoot,  $O_p = (O/V) \times 100$ , is a relative stability criterion, with 10% - 20% as an acceptable value.

Closed-Loop  
Response  
of C to a Step of V  
when U = 0



- Settling time,  $T_s$ , the time it takes for the response to get and stay within a specified percentage, e.g., 5%, of  $V$ , combines stability and speed of response aspects.
- The decay ratio, the ratio of the second overshoot divided by the first, is a relative stability criterion used most often in the process control industry, with 1/4 a common design value.

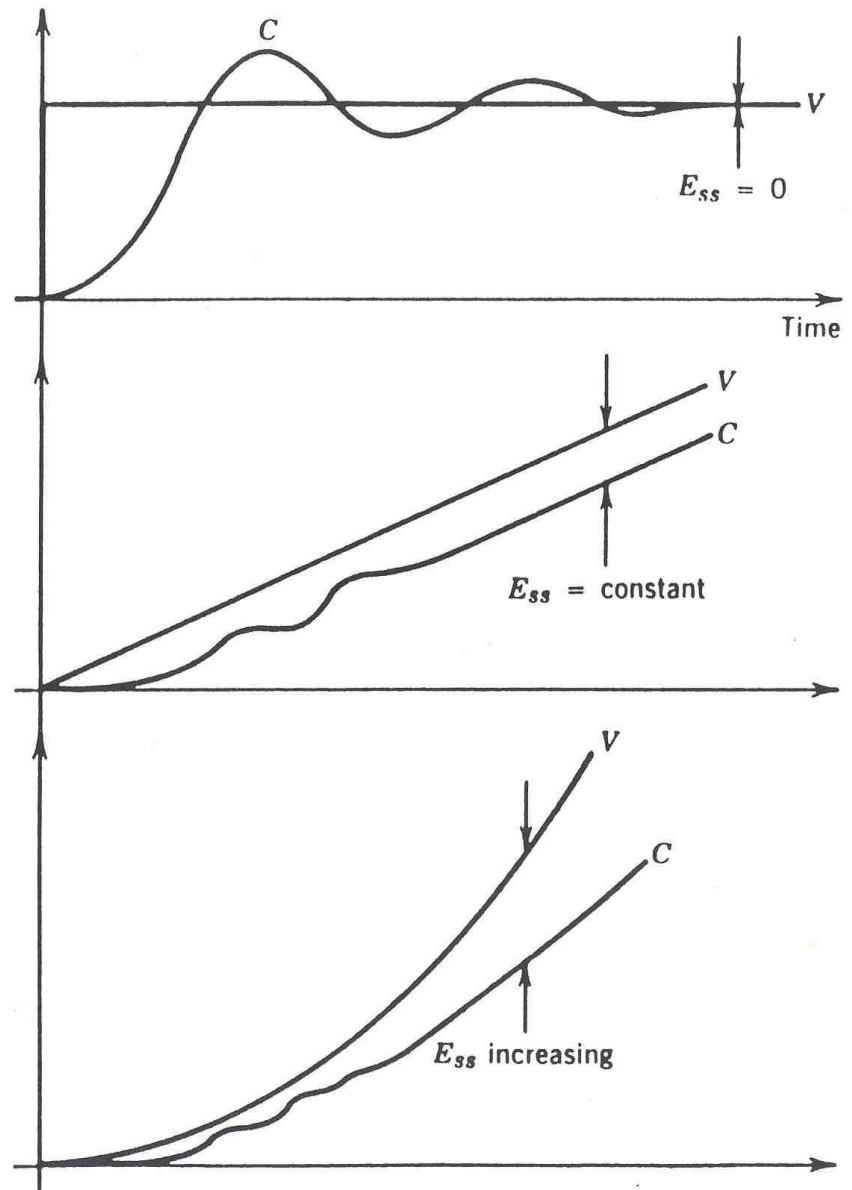
Which System is  
Faster?  
A or B?



- Certain math models of systems will predict, for given commands or disturbances, steady-state errors that are precisely zero, but no real system can achieve this perfection.
- Nonzero errors are always present because of nonlinearities, measurement uncertainties, etc.
- To determine the steady-state error set up the closed-loop system differential equation in which error ( $V-C$ ) is the unknown. Solution of this equation gives a transient solution that always decays to zero for an absolutely stable system.

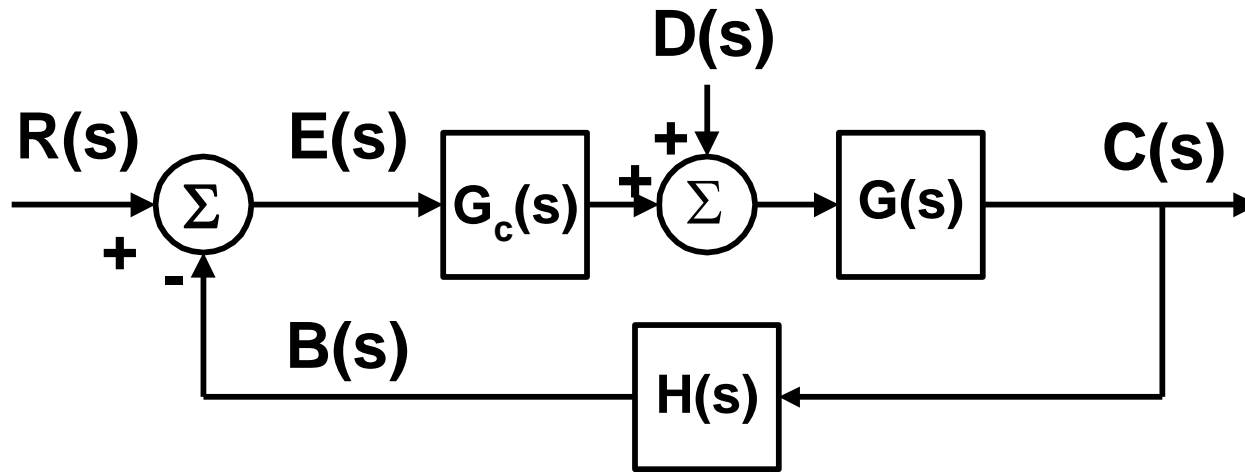
- The remaining solution is, by definition, the steady-state error, whether it is itself steady or time varying. That is, steady-state error need not be a constant value.
- The steady-state error,  $E_{ss}$ , depends on both the system and the input command or disturbance that causes the error.
- There is a certain pattern of behavior as the input is made more difficult from the steady-state viewpoint. This type of pattern can be expected for both commands and disturbances in all linear systems, though details will vary.

Effect of  
Command Severity  
on Steady-State  
Error





- Steady-State Error



- Assume  $H(s) = 1$  and  $D(s) = 0$ . The error is then  $E(s)$  which equals  $R(s) - C(s)$ .

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G_c(s)G(s)}$$

$$E(s) = \frac{R(s)}{1 + G_c(s)G(s)}$$

$$e_{ss}(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{R(s)}{1 + G_c(s)G(s)}$$

- The final value theorem, assuming closed-loop stability, tells us that:

$$e_{ss}(t) = \lim_{s \rightarrow 0} sE(s)$$

- We are interested in the steady-state error for step, ramp, and parabolic inputs, i.e.,

$$R(s) = \frac{1}{s^{n+1}} \quad n = 0, 1, 2$$

- Therefore

$$e_{ss}(t) = \lim_{s \rightarrow 0} \frac{1}{s^n + s^n G_1(s)G_2(s)}$$

– Step Input:  $R(s) = 1/s$

$$e_{ss}(t) = \lim_{s \rightarrow 0} s \frac{\frac{1}{s}}{1 + G_c(s)G(s)} = \lim_{s \rightarrow 0} \frac{1}{1 + G_c(s)G(s)} = \frac{1}{1 + K_p}$$

$$K_p \equiv \lim_{s \rightarrow 0} G_c(s)G(s)$$

– Ramp Input:  $R(s) = 1/s^2$

Static Error Constants:  
 $K_p$  and  $K_v$

$$e_{ss}(t) = \lim_{s \rightarrow 0} s \frac{\frac{1}{s^2}}{1 + G_c(s)G(s)} = \lim_{s \rightarrow 0} \frac{1}{s + sG_c(s)G(s)}$$

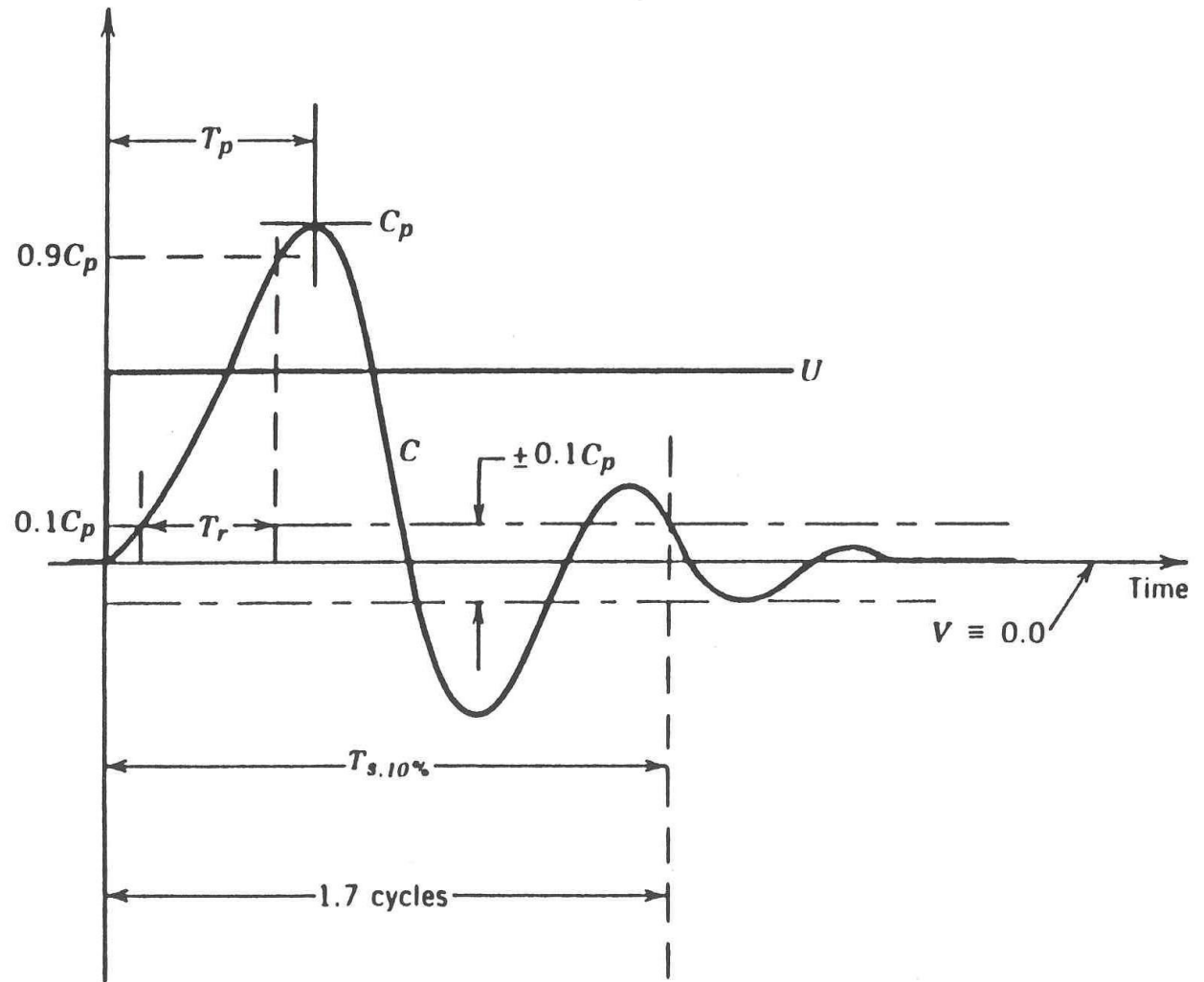
$$= \lim_{s \rightarrow 0} \frac{1}{sG_c(s)G(s)} = \frac{1}{K_v}$$

$$K_v \equiv \lim_{s \rightarrow 0} sG_c(s)G(s)$$

- System type is the order of the input polynomial that the closed-loop system can track with finite error.
- For example, if  $G_1(s)G_2(s)$  has no poles at the origin, the closed-loop is a Type 0 system and can track a constant with finite steady-state error. A Type 1 system (one pole at the origin) can track a constant with zero error and a ramp with finite error. A Type 2 system (two poles at the origin) can track both a constant and ramp with zero error and a parabola with finite error.

- When system input is a disturbance  $U$  ( $V=0$ ) some of these criteria can still be applied, although others cannot.
- It is still possible to define a peak time  $T_p$ , however  $T_r$ ,  $T_s$ , and  $O_p$  are all referenced to step size  $V$ , which is now zero, thus they cannot be used.
- One possibility is to use peak value  $C_p$  as a reference value to define  $T_r$  and  $T_s$ .
- To replace  $O_p$  as a stability specification one could use the decay ratio defined earlier or perhaps the number of cycles to damp the amplitude to say, 10% of  $C_p$ . The smaller the number of cycles, the better the stability.
- Definition of steady-state error still applies and we would again expect the same trend of worsening error as  $U$  changed from step to ramp to parabola.

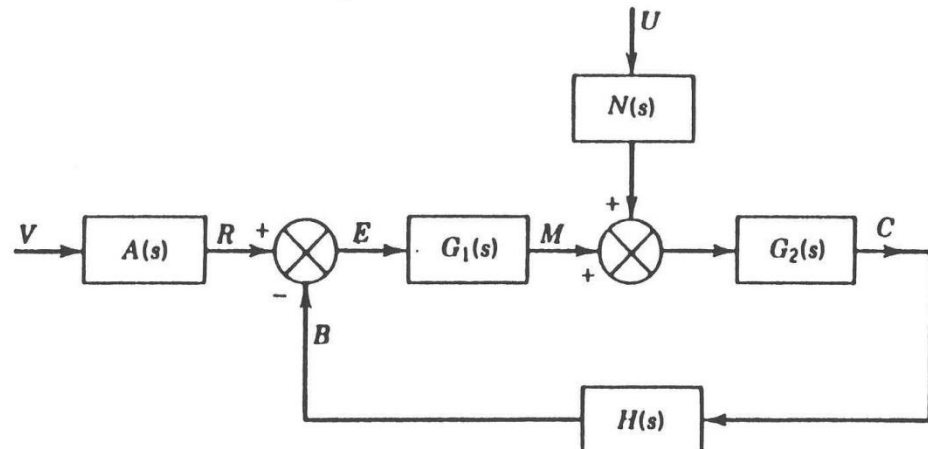
Time-Domain  
Performance  
Specifications  
for a Disturbance  
Input



# Frequency Response Performance Specifications

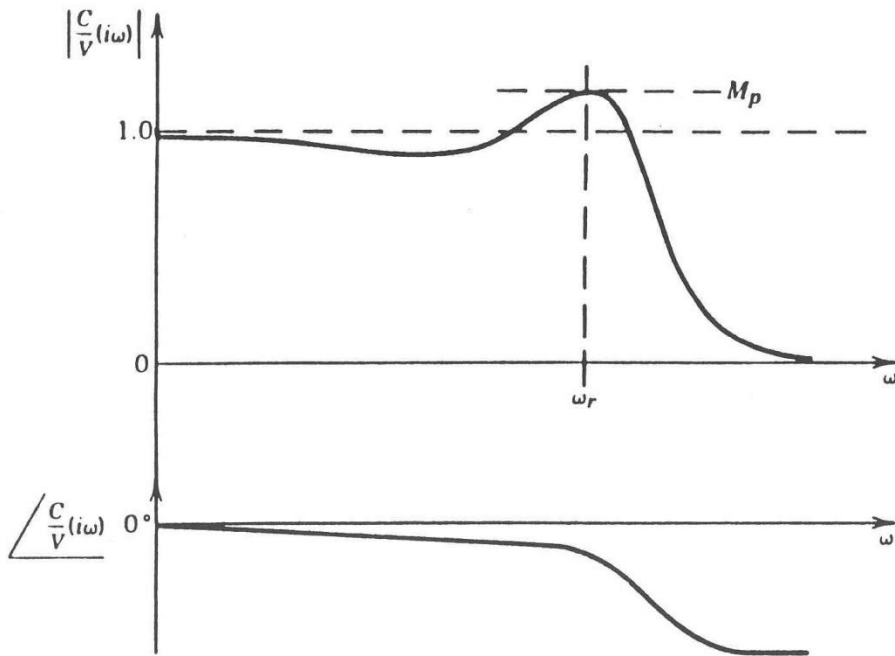
- Let  $V$  be a sine wave ( $U = 0$ ) and wait for transients to die out.
- Every signal will be a sine wave of the same frequency. We can then speak of amplitude ratios and phase angles between various pairs of signals.

$$\frac{C}{V}(i\omega) = \frac{AG_1G_2(i\omega)}{1 + G_1G_2H(i\omega)}$$

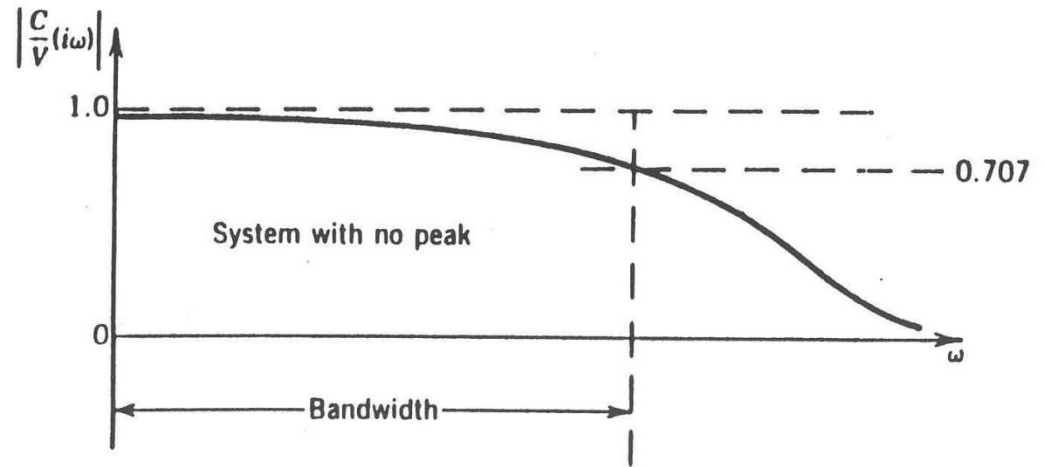


- The most important pair involves  $V$  and  $C$ . Ideally  $(C/V)(i\omega) = 1.0$  for all frequencies.
- Amplitude ratio and phase angle will approximate the ideal values of 1.0 and 0 degrees for some range of low frequencies, but will deviate at higher frequencies.





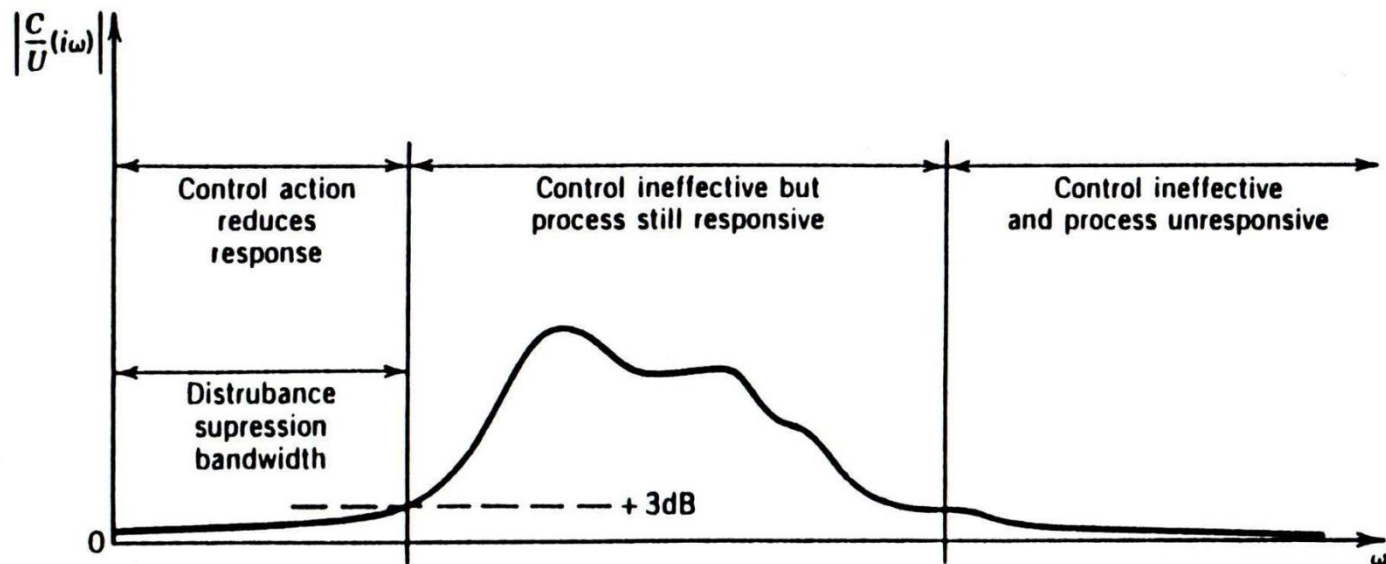
Typical Closed-Loop  
 Frequency Response  
 Curves



- The frequency at which a resonant peak occurs,  $\omega_r$ , is a speed of response criterion. The higher  $\omega_r$ , the faster the system response.
- The peak amplitude ratio,  $M_p$ , is a relative stability criterion. The higher the peak, the poorer the relative stability. If no specific requirements are pushing the designer in one direction or the other,  $M_p = 1.3$  is often used as a compromise between speed and stability.
- For systems that exhibit no peak, the bandwidth is used for a speed of response specification. The bandwidth is the frequency at which the amplitude ratio has dropped to 0.707 times its zero-frequency value. It can of course be specified even if there is a peak.

- If we set  $V = 0$  and let  $U$  be a sine wave, we can measure or calculate  $(C/U)(i\omega)$  which should ideally be 0 for all frequencies. A real system cannot achieve this perfection but will behave typically as shown.

### Closed-Loop Frequency Response to a Disturbance Input



- Two open-loop performance criteria in common use to specify relative stability are gain margin and phase margin.
- The open-loop frequency response is defined as  $(B/E)(i\omega)$ . One could open the loop by removing the summing junction at R, B, E and just input a sine wave at E and measure the response at B. This is valid since  $(B/E)(i\omega) = G_1G_2H(i\omega)$ . Open-loop experimental testing has the advantage that open-loop systems are rarely absolutely unstable, thus there is little danger of starting up an untried apparatus and having destructive oscillations occur before it can be safely shut down.
- The utility of open-loop frequency-response rests on the Nyquist stability criterion.

- Gain margin (GM) and phase margin (PM) are in the nature of safety factors such that  $(B/E)(i\omega)$  stays far enough away from  $1 \angle -180^\circ$  on the stable side.
- Gain margin is the multiplying factor by which the steady state gain of  $(B/E)(i\omega)$  could be increased (nothing else in  $(B/E)(i\omega)$  being changed) so as to put the system on the edge of instability, i.e.,  $(B/E)(i\omega)$  passes exactly through the -1 point. This is called marginal stability.
- Phase margin is the number of degrees of additional phase lag (nothing else being changed) required to create marginal stability.
- Both a good gain margin and a good phase margin are needed; neither is sufficient by itself.

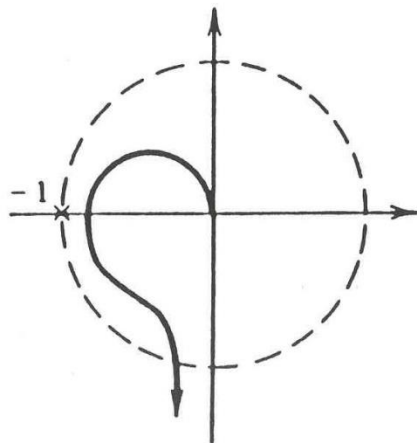
# Open-Loop Performance Criteria: Gain Margin and Phase Margin

A system must have adequate stability margins.

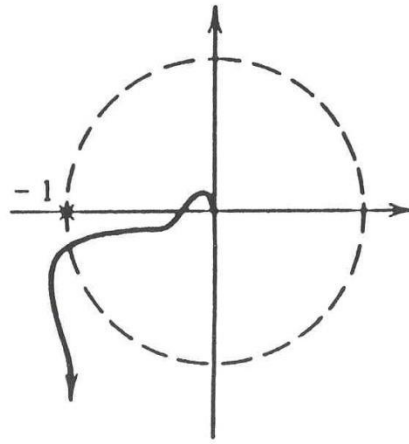
Both a good gain margin and a good phase margin are needed.

Useful lower bounds:

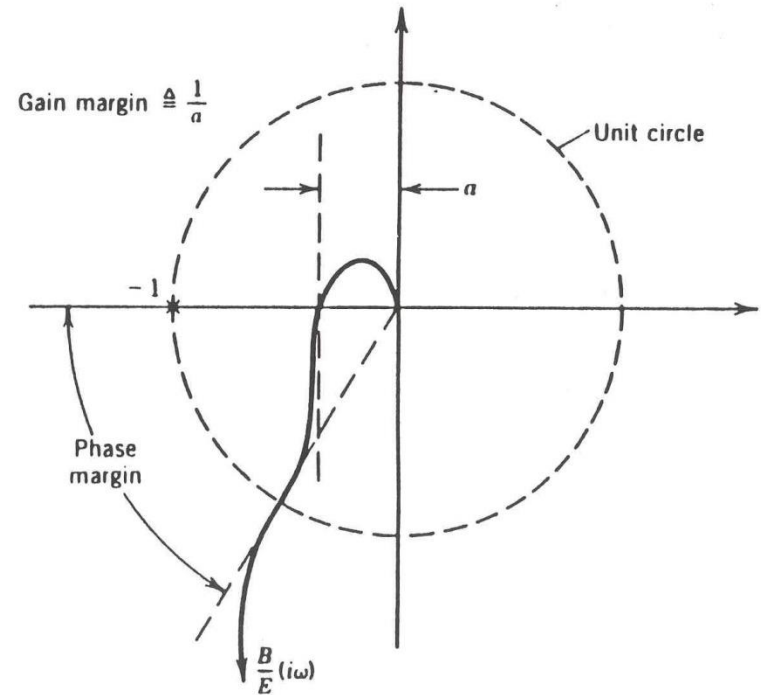
$$GM > 2.5 \quad PM > 30^\circ$$



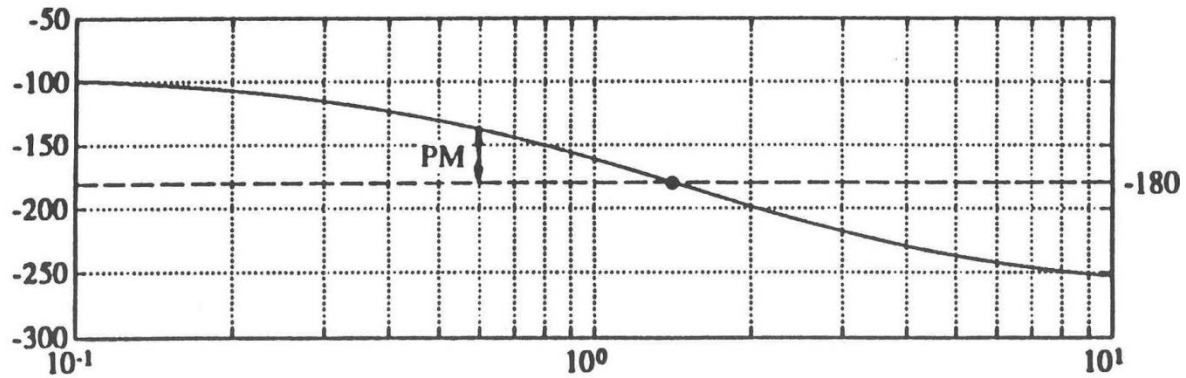
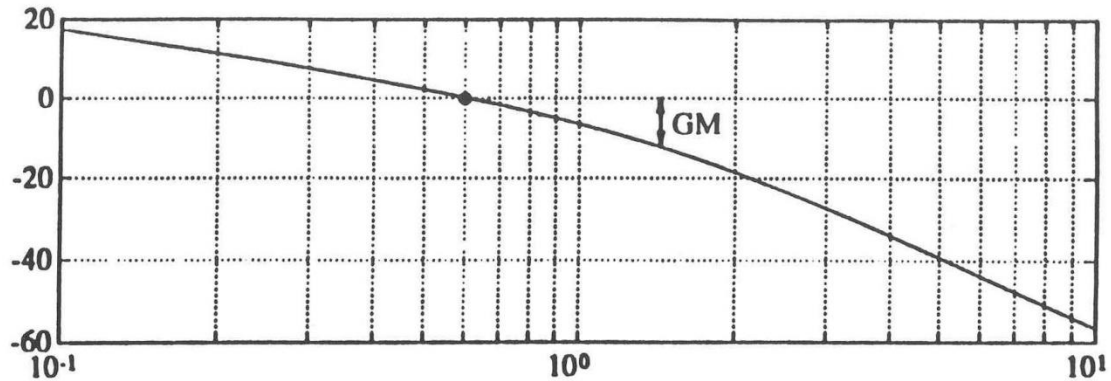
Good phase margin,  
poor gain margin



Good gain margin,  
poor phase margin



# Bode Plot View of Gain Margin and Phase Margin



- It is important to realize that, because of model uncertainties, it is not merely sufficient for a system to be stable, but rather it must have adequate stability margins.
- Stable systems with low stability margins work only on paper; when implemented in real time, they are frequently unstable.
- The way uncertainty has been quantified in classical control is to assume that either gain changes or phase changes occur. Typically, systems are destabilized when either gain exceeds certain limits or if there is too much phase lag (i.e., negative phase associated with unmodeled poles or time delays).
- As we have seen these tolerances of gain or phase uncertainty are the gain margin and phase margin.



- Consider the following design problem: Given a plant transfer function  $G_2(s)$ , find a compensator transfer function  $G_1(s)$  which yields the following:
  - stable closed-loop system
  - good command following
  - good disturbance rejection
  - insensitivity of command following to modeling errors (performance robustness)
  - stability robustness with unmodeled dynamics
  - sensor noise rejection

- Without closed-loop stability, a discussion of performance is meaningless. It is critically important to realize that the compensator  $G_1(s)$  is actually designed to stabilize a nominal open-loop plant  $G_2^*(s)$ . Unfortunately, the true plant is different from the nominal plant due to unavoidable modeling errors, denoted by  $\delta G_2(s)$ . Thus the true plant may be represented by

$$G_2(s) = G_2^*(s) + \delta G_2(s)$$

- Knowledge of  $\delta G_2(s)$  should influence the design of  $G_1(s)$ . We assume here that the actual closed-loop system, represented by the true closed-loop transfer function is absolutely stable.

(unity feedback assumed)

$$\frac{G_1(s) \left[ G_2^*(s) + \delta G_2(s) \right]}{1 + G_1(s) \left[ G_2^*(s) + \delta G_2(s) \right]}$$

## Desired Shape for Open-Loop Transfer Function

