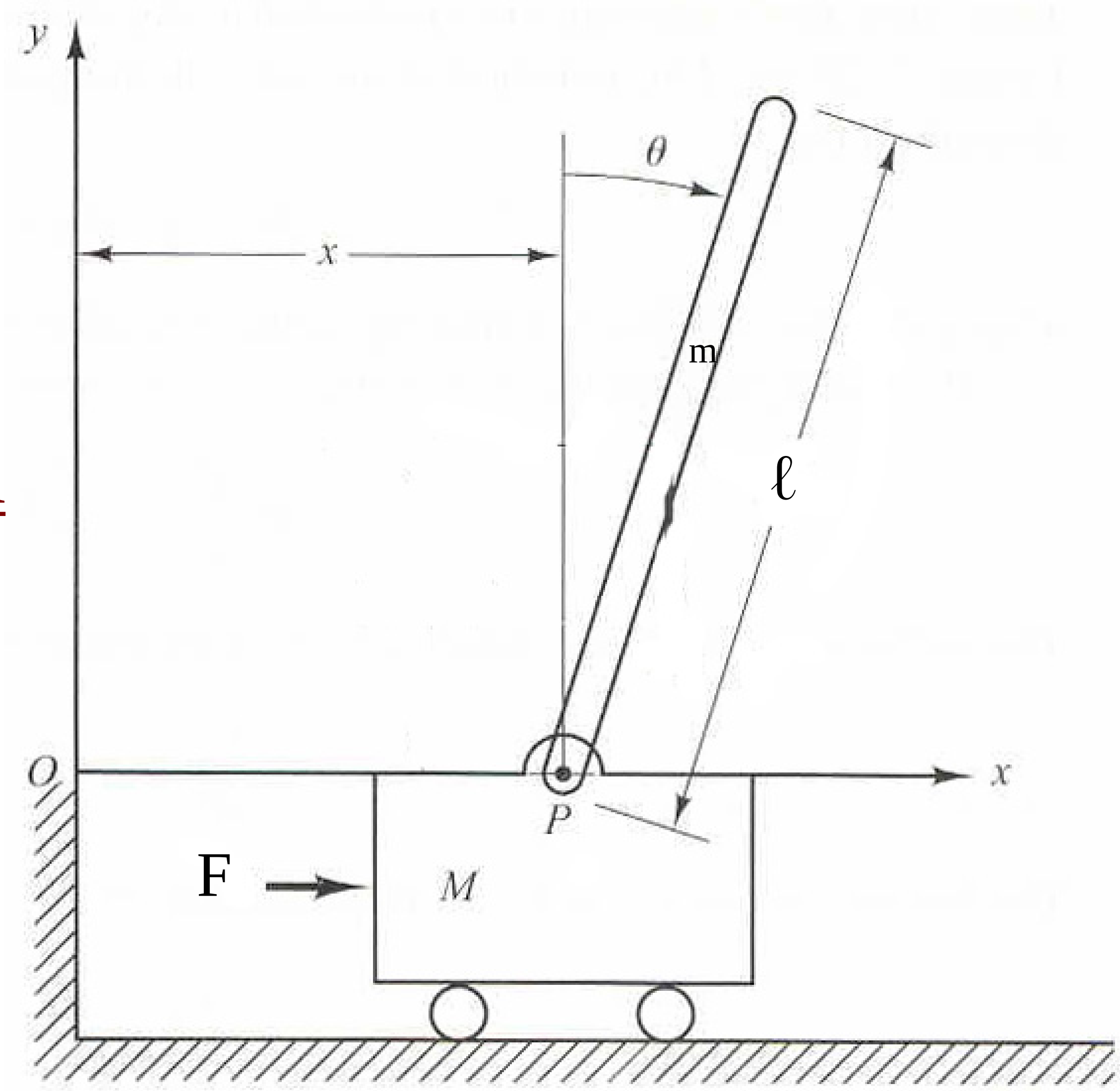
Inverted Pendulum System



– Shown is a physical model of an inverted pendulum on a cart driven by an actuator. The only sensors are the angular position sensor for the pendulum and the position sensor for the cart. The inverted pendulum is unstable in that it may fall over at any time unless a suitable control force F is applied to the cart. The physical model is based on the following assumptions:
  - Only two-dimensional motion is possible.
  - Inverted pendulum pivot is frictionless.
  - Dynamic response of the actuator and sensors are sufficiently fast that they can be considered instantaneous.
  - Neglect external disturbance forces.
  - Neglect friction between the cart and the ground.
  - The inverted pendulum is a rigid, homogeneous, thin rod of length l and mass m.
  - Cart is rigid with mass M.

– Requirements:

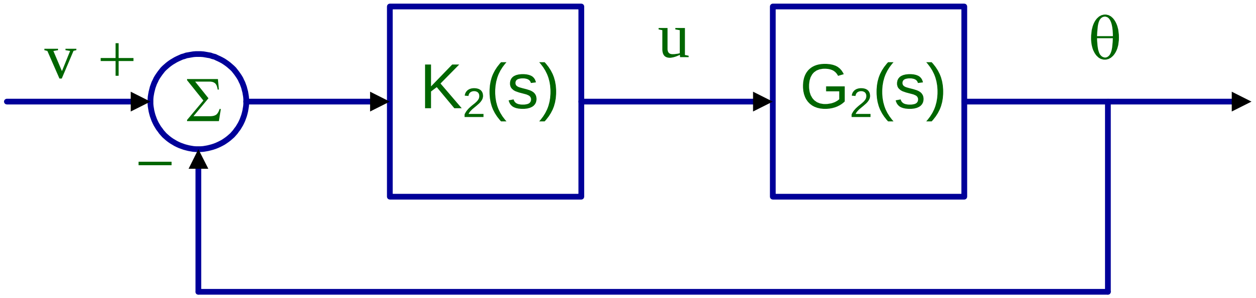
  - Draw a FBD of the inverted pendulum system and derive the mathematical model, i.e., the equations of motion, for the system. Note that these equations are nonlinear.
  - Linearize the equations of motion about the operating point $\theta = 0^o$. Represent them in state-space form.
  - Analytically derive the transfer functions θ / F and x / F.
  - The physical parameters for this system are: *M = 1 kg*, *m = 0.1 kg*, *l = 1 meter*. Using MatLab and/or LabVIEW, these parameter values, and your state-space equations, validate your analytically-derived transfer functions.

- Using MatLab and/or LabVIEW design a classical control system to simultaneously stabilize the inverted pendulum position and maintain the cart position at its starting point, in response to some non-zero initial conditions, e.g., $\theta = 0.15\ rad$, $d\theta/dt = 0\ rad/sec$, $x = 0\ meters$, $dx/dt = 0\ m/s$. Describe the performance of your design in terms of frequency-domain and time-domain parameters. Support your design with appropriate root-locus, frequency-response and time-response plots. How sensitive is your design to changes in plant parameters?
- Design an optimal full-state feedback controller to both balance the pendulum and center the cart. Vary the weightings on the states and control effort and compare performances.
- Compare the two control design approaches: classical and state-space.

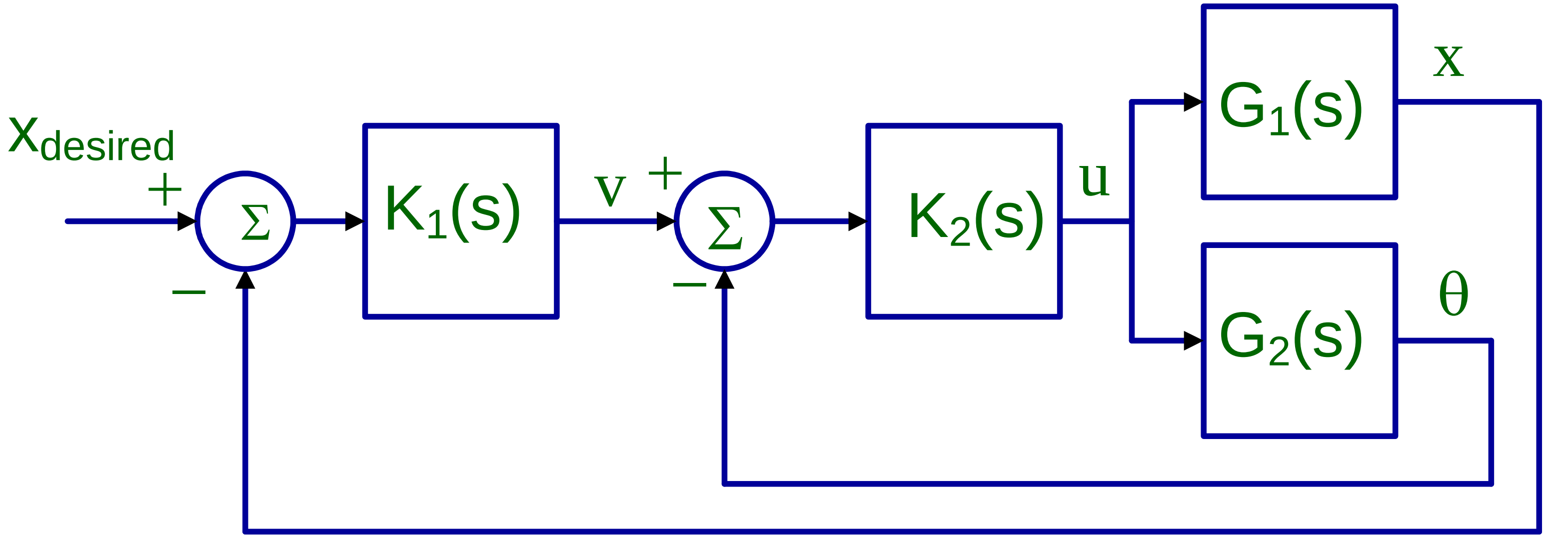
- <u>Supplement for Inverted Pendulum Classical Control Design</u>

$$G_1(s) = \frac{N_1}{D} = \frac{x}{u}$$

$$G_2(s) = \frac{N_2}{D} = \frac{\theta}{u}$$



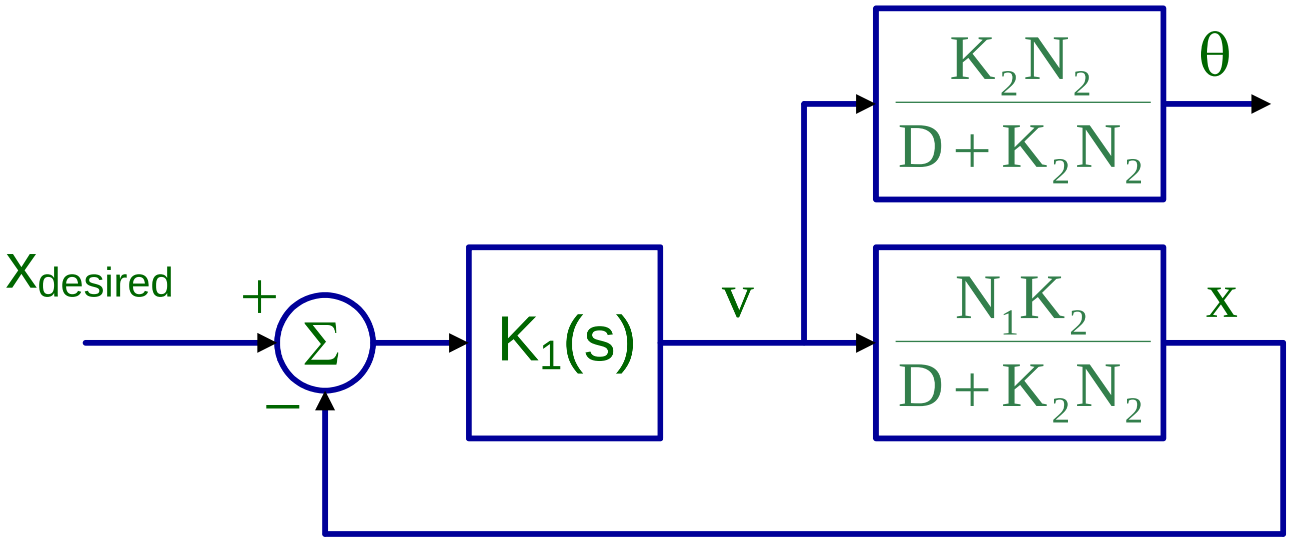
$$\frac{\theta}{v} = \frac{K_2 G_2}{1 + K_2 G_2} = \frac{K_2 \dfrac{N_2}{D}}{1 + \dfrac{K_2 N_2}{D}} = \frac{K_2 N_2}{D + K_2 N_2}$$



$$u = -K_2\theta + K_2 v$$

$$x = G_1 u = \frac{N_1}{D}\left(-K_2\theta + K_2 v\right) = \frac{N_1}{D}\left(-K_2\frac{K_2 N_2}{D + K_2 N_2}v + K_2 v\right)$$

$$= v\left[\frac{N_1 K_2}{D + K_2 N_2}\right]$$



$$\frac{x}{x_{desired}} = \frac{\dfrac{K_1K_2N_1}{D+K_2N_2}}{1+\dfrac{K_1K_2N_1}{D+K_2N_2}} = \frac{K_1K_2N_1}{D+K_2N_2+K_1K_2N_1}$$

- <u>Answers</u>
  - Nonlinear Equations of Motion

$$\left(M+m\right)\ddot{x}+\left(\frac{m\ell}{2}\cos\theta\right)\ddot{\theta}-\frac{m\ell}{2}\dot{\theta}^2\sin\theta=F$$

$$\left(\frac{1}{3}m\ell^2\right)\ddot{\theta}+\left(\frac{m\ell}{2}\cos\theta\right)\ddot{x}-\frac{mg\ell}{2}\sin\theta=0$$

  - Linearized Equations of Motion

$$\left(M+m\right)\ddot{x}+\left(\frac{m\ell}{2}\right)\ddot{\theta}=F$$

$$\left(\frac{1}{3}m\ell^2\right)\ddot{\theta}+\left(\frac{m\ell}{2}\right)\ddot{x}-\frac{mg\ell}{2}\theta=0$$

– State-Space Equations

$$q_1 = x \qquad q_2 = \dot{x} \qquad q_3 = \theta \qquad q_4 = \dot{\theta}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{6(M+m)g}{\ell(4M+m)} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{4}{4M+m} \\ 0 \\ \dfrac{-6}{\ell(4M+m)} \end{bmatrix} [F]$$

– Transfer Functions

$$\frac{\Theta}{F}(s)=\frac{-\frac{1}{2}s^2}{s^2\left[\left(\frac{M}{3}+\frac{m}{12}\right)\ell s^2-\left(\frac{M+m}{2}\right)g\right]}$$

$$\frac{X}{F}(s)=\frac{\frac{\ell}{3}s^2-\frac{g}{2}}{s^2\left[\left(\frac{M}{3}+\frac{m}{12}\right)\ell s^2-\left(\frac{M+m}{2}\right)g\right]}$$

– State-Space Equations and Transfer Functions (with parameter values)

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -0.7178 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 15.7917 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.9756 \\ 0 \\ -1.4634 \end{bmatrix} [F]$$

$$\frac{\Theta}{F}(s) = \frac{-1.4634s^2}{s^2\left(s^2 - 15.7917\right)} \qquad \frac{X}{F}(s) = \frac{0.9756s^2 - 14.3560}{s^2\left(s^2 - 15.7917\right)}$$

– Sample Controllers

$$K_2(s) = -155\frac{s+5}{s+20}$$

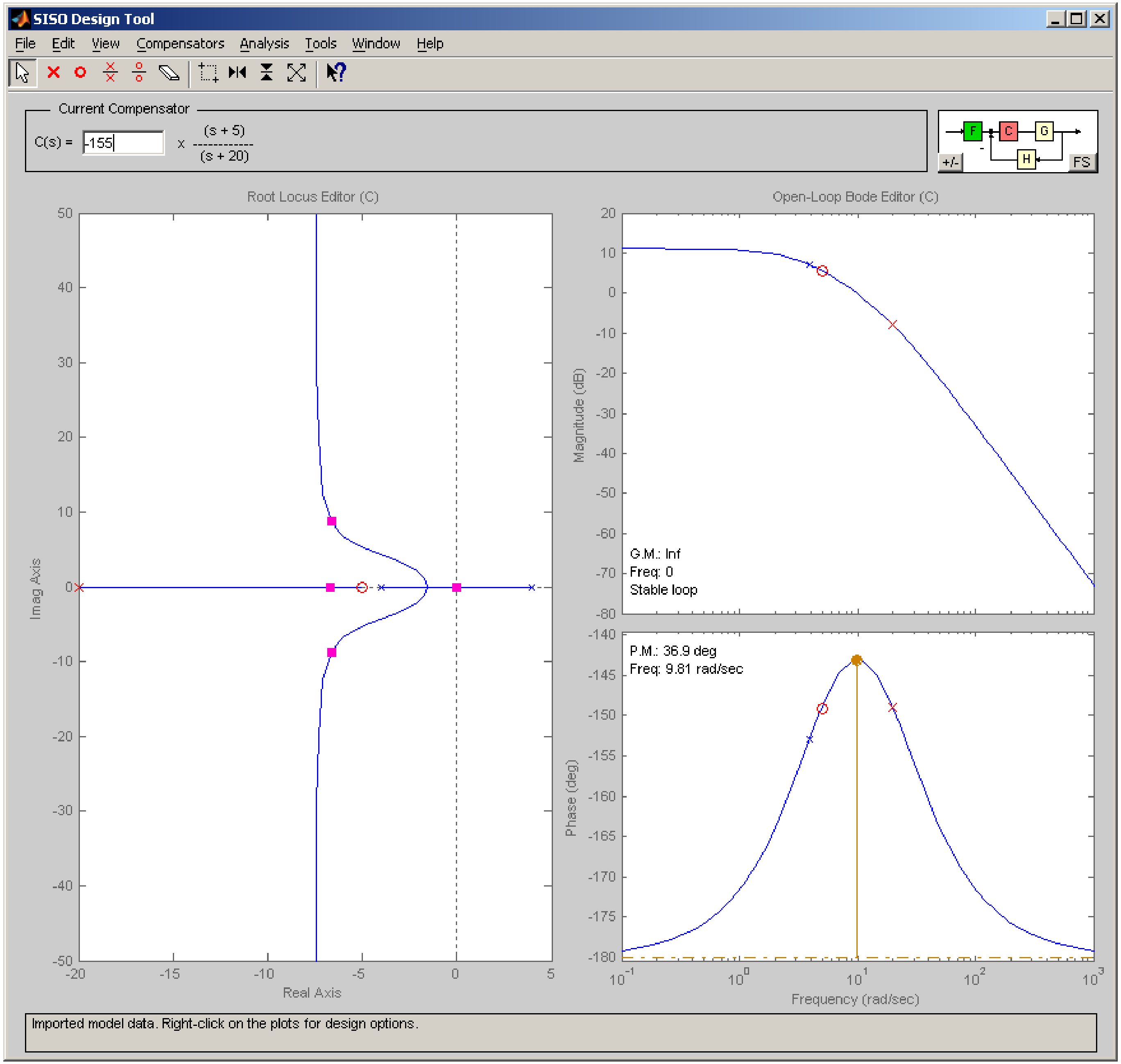
$$K_1(s) = 0.28\frac{s+0.25}{s+4}$$

Translational IP KC 2005

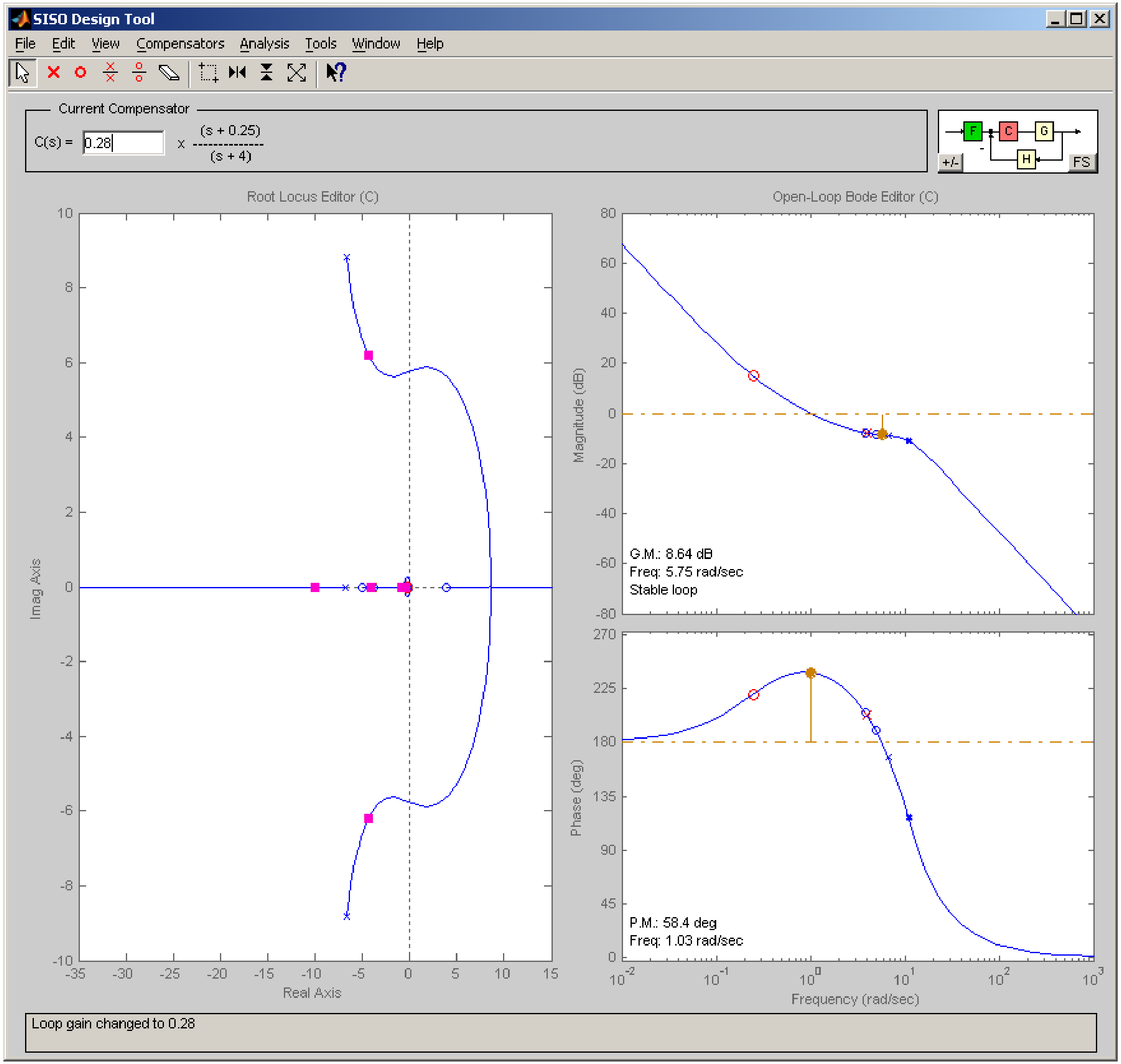
```
N2=-1.4634*[1 0 0];
N1=[0.9756 0 -14.3560];
Den=[1 0 -15.7917 0 0];
plant_1=tf(N2,Den);
A=[0 1 0 0;0 0 -0.7178 0;0 0 0 1;0 0 15.7917 0];
B=[0 0.9756 0 -1.4634]';
C=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
D=[0 0 0 0]';
%K2Gain=-37.24;
%K2Num=[1 3.27];
%K2Den=[1 7.64];
K2Gain=-155;
K2Num=[1 5];
K2Den=[1 20];
K2=K2Gain*tf(K2Num,K2Den);
plant_2_num=K2Gain*conv(N1,K2Num);
Q1=conv(Den,K2Den);
Q2=conv(K2Gain*K2Num,N2);
plant_2_den=[Q1(1,1) Q1(1,2) Q1(1,3)+Q2(1,1) Q1(1,4)+Q2(1,2) Q1(1,5)+Q2(1,3) Q1(1,6)+Q2(1,4)];
plant_2=tf(plant_2_num,plant_2_den);
%K1Gain=0.16;
%K1Num=[1 0.25];
%K1Den=[1 3.84];
K1Gain=.28;
K1Num=[1 .25];
K1Den=[1 4];
```

MatLab M-File

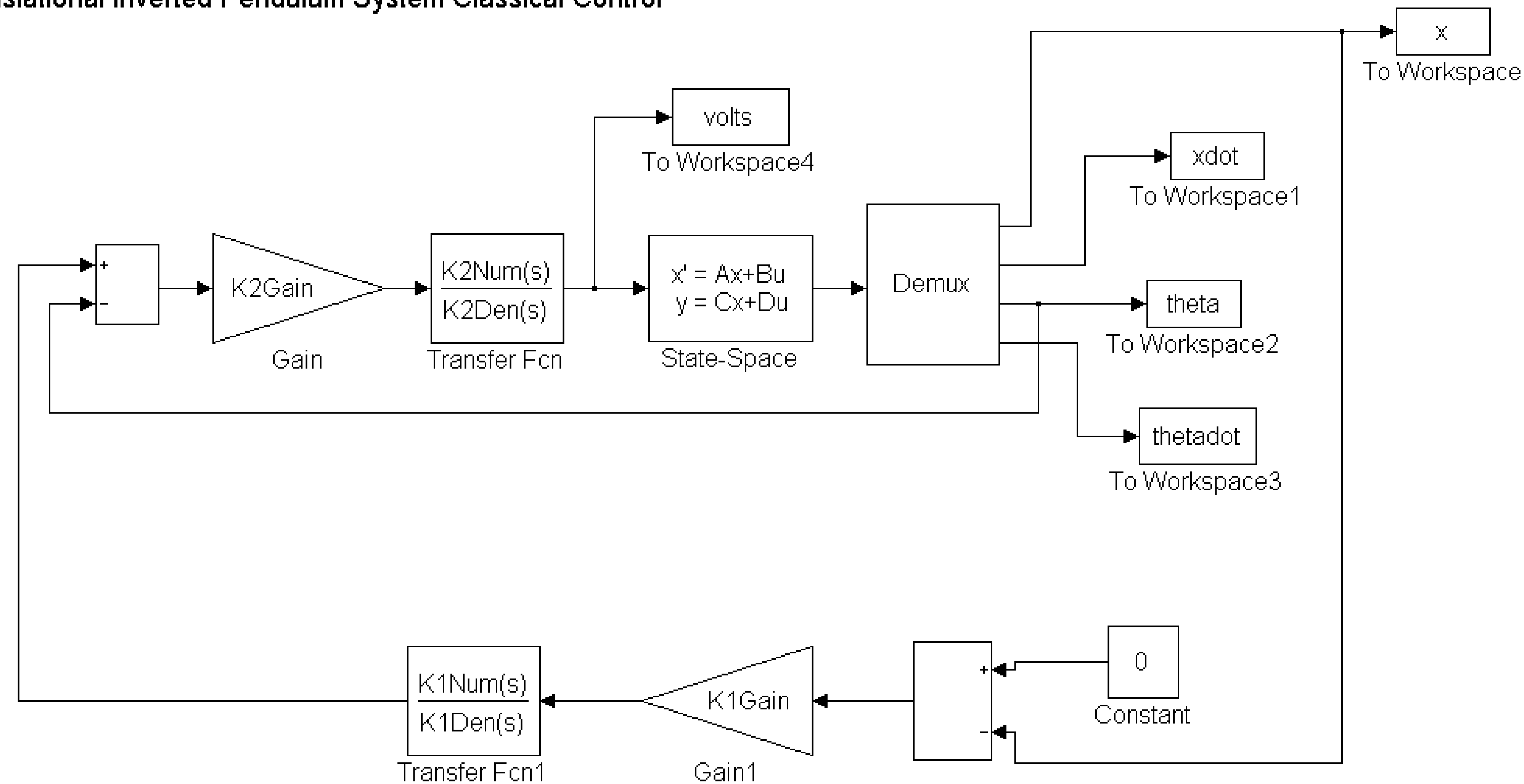
Design of Inner-Loop Controller

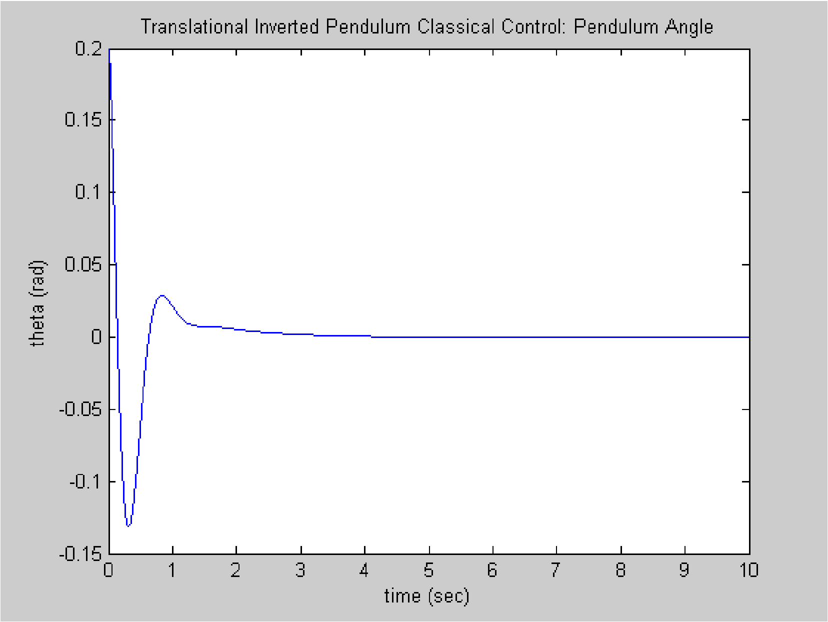


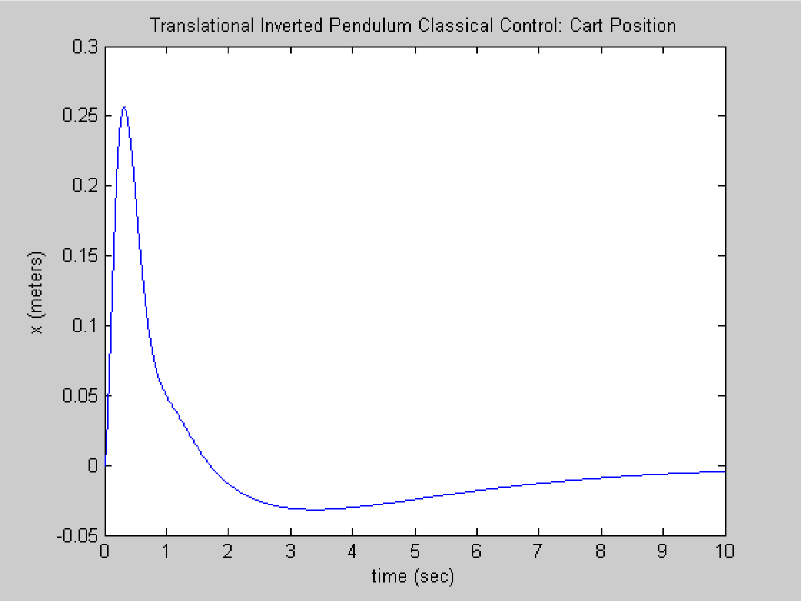
Design of Outer-Loop Controller

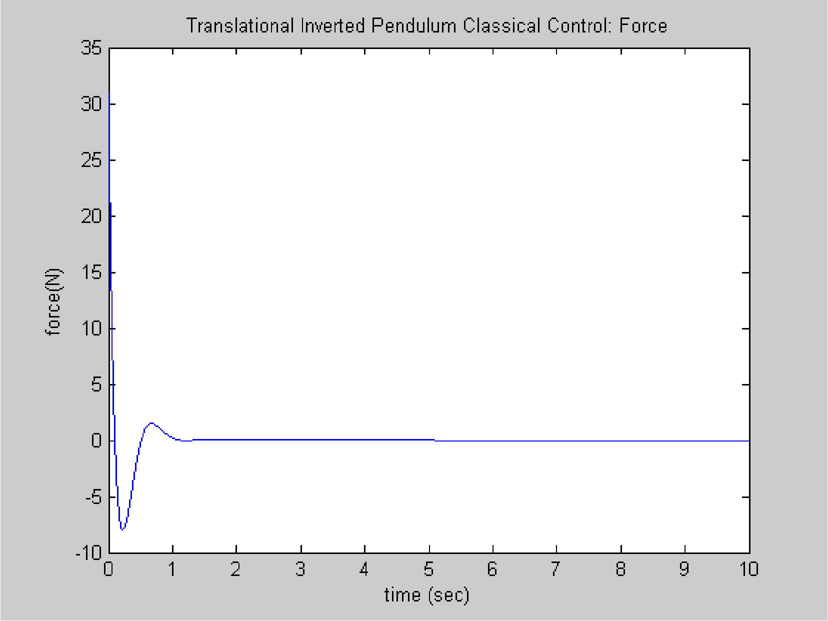


Translational Inverted Pendulum System Classical Control

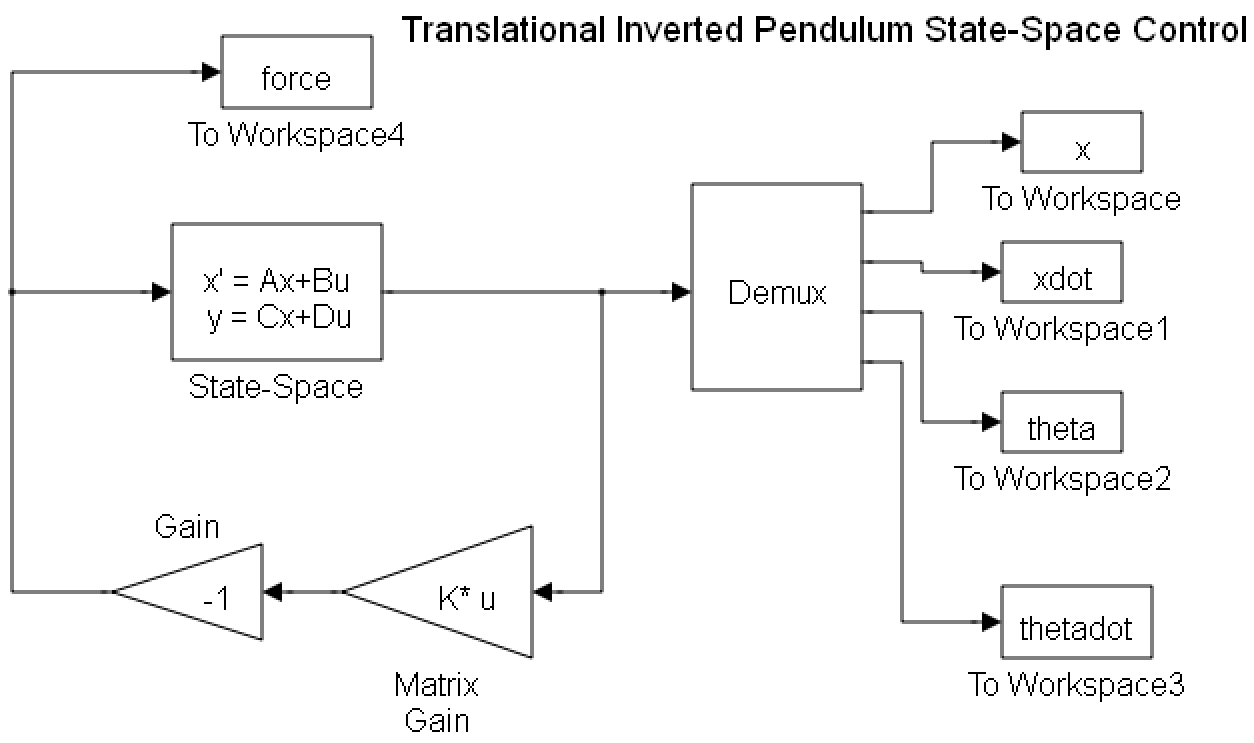


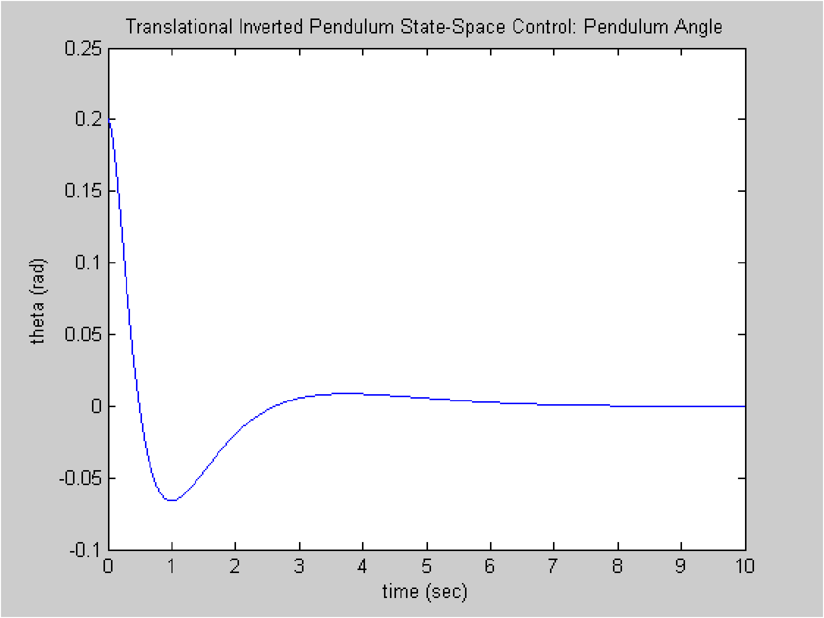

Translational Inverted Pendulum Classical Control: Pendulum Angle
theta (rad)
time (sec)
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
0
1
2
3
4
5
6
7
8
9
10

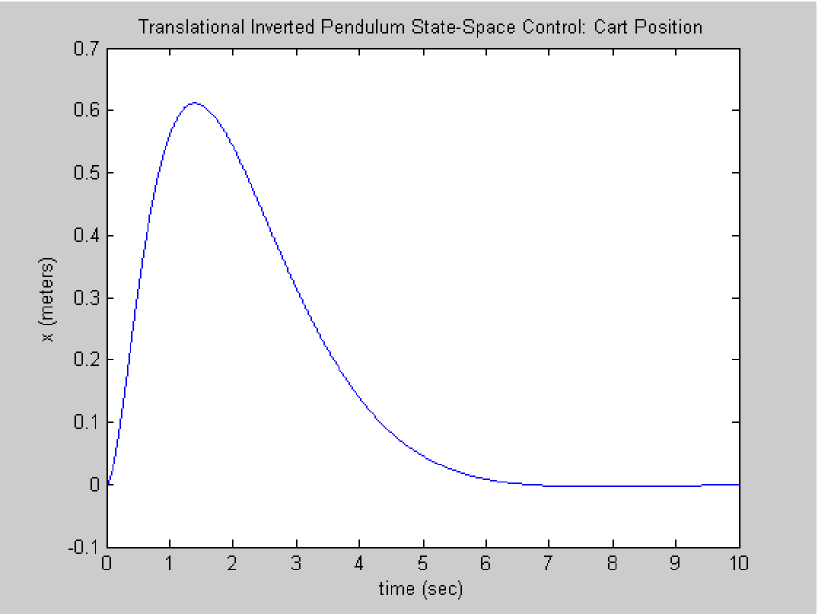

Translational Inverted Pendulum Classical Control: Cart Position
x (meters)
time (sec)
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
1
2
3
4
5
6
7
8
9
10

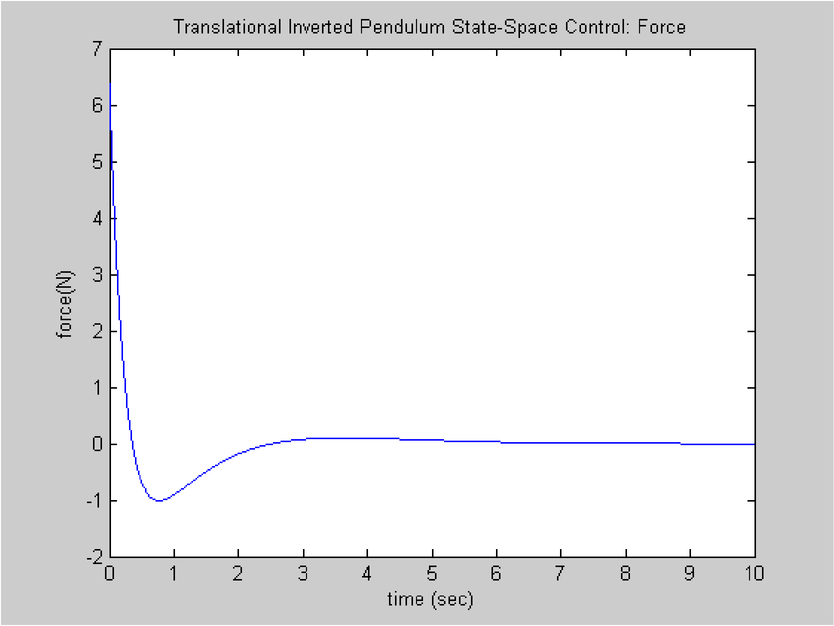

Translational Inverted Pendulum Classical Control: Force
force(N)
time (sec)
35
30
25
20
15
10
5
0
-5
-10
0
1
2
3
4
5
6
7
8
9
10

Q = [1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
R = [1];
[K,S,E] = lqr(A,B,Q,R);

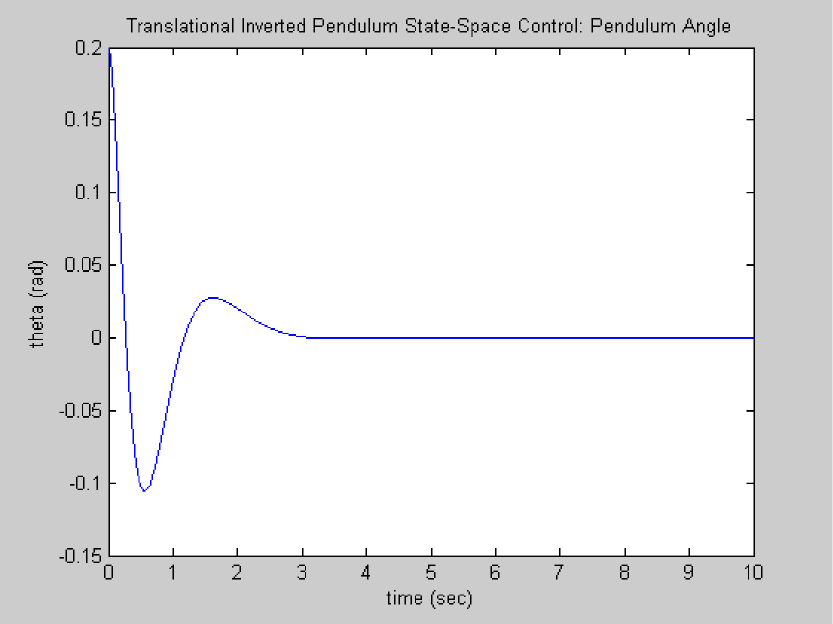


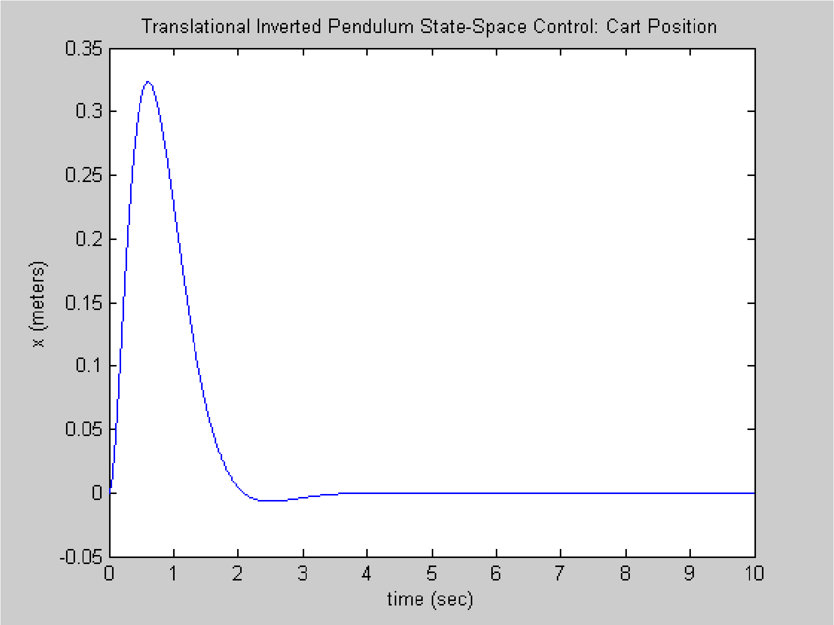


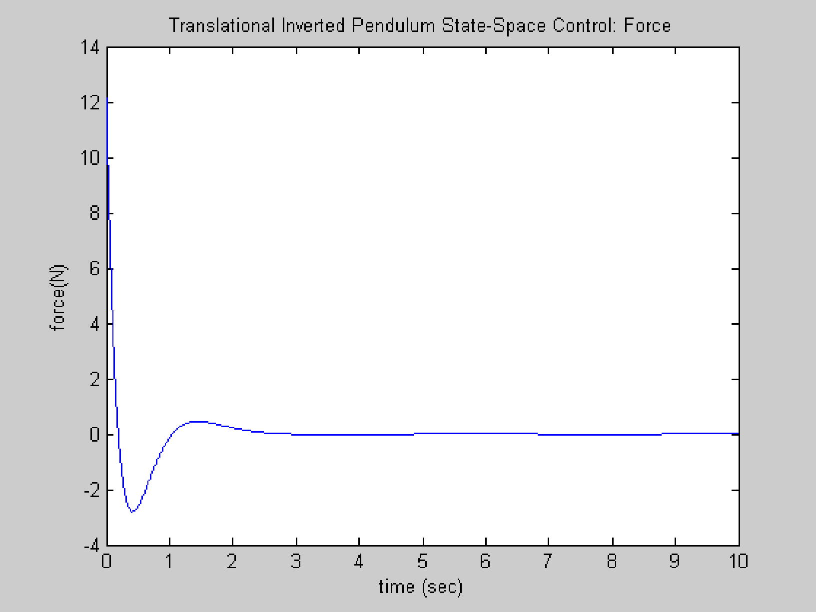




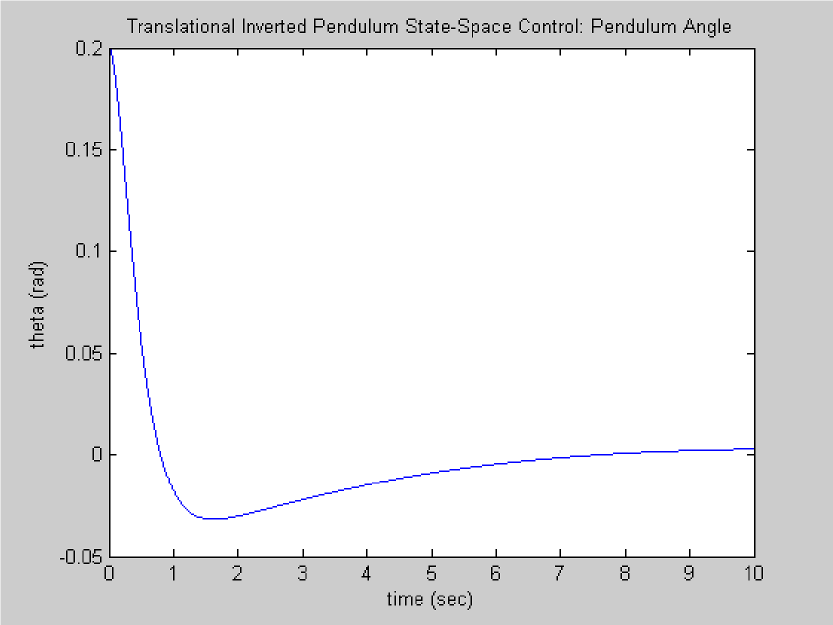
$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R = [1]$$

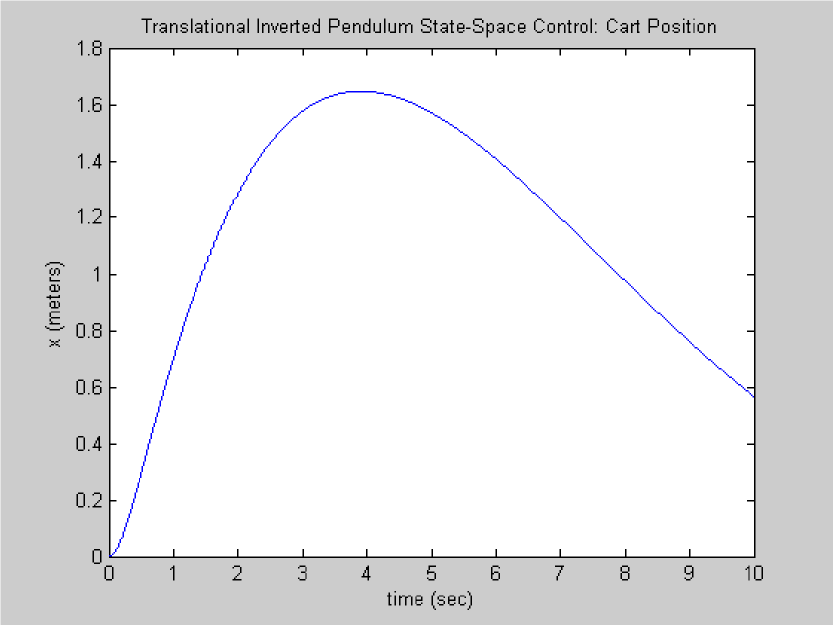


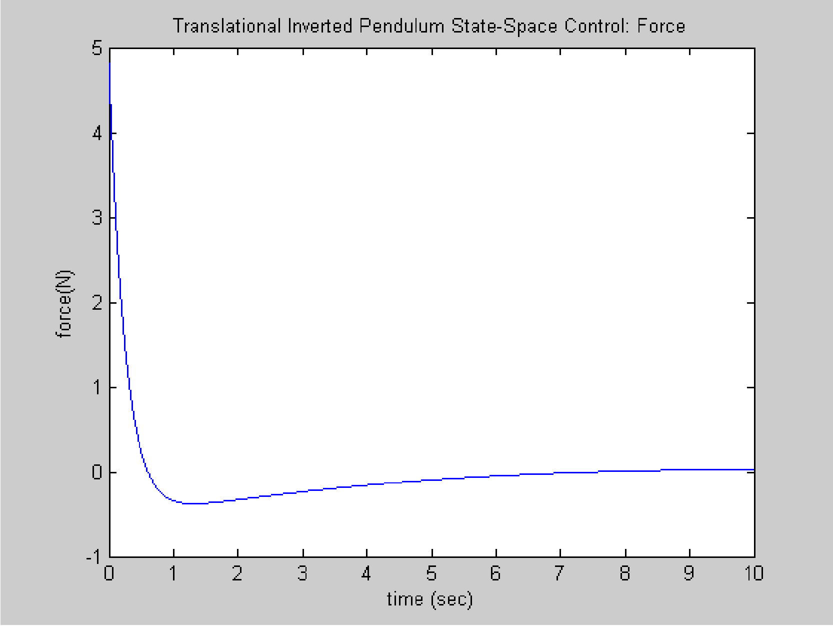




$$Q = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R = [1]$$







$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R = [100]$$