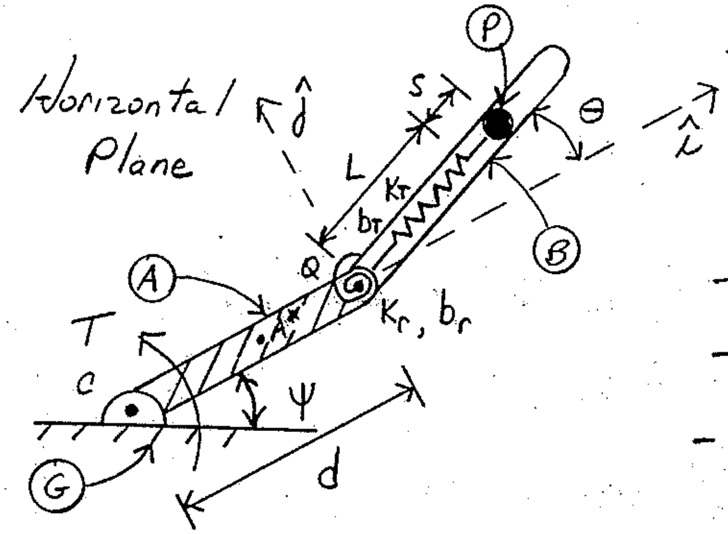
DYNAMICS AND CONTROL OF MULTIBODY SYSTEMS

K. Craig

Problem Statement:



- A is a rigid link, mass M, and mass moment of inertia about point C is I_c.
- B is a massless tube.
- P is a particle, mass m.
- C and Q are frictionless revolute joints.

- A torsional spring and damper are at revolute joint Q, with constants K_r and b_r, respectively. The torsional spring is unstretched when $\theta = 0$.
- A translational spring connects the particle P to the base of the tube. Its constant is K_T and the spring is unstretched when the distance from Q to P is equal to L. Viscous damping exists within the tube, with constant b_T.
- A torque T is applied to the rigid link A.
- This model is very similar to a beam, having extensional and bending flexibility, connected to a rigid hub.

Derivation of Equations of Motion

A) Kane's Method

(1) Choose generalized speeds:

System has 3 DOF with generalized coordinates s, θ, ψ.

Define generalized speeds as:

$$u_1 = \dot{s} \qquad u_2 = \dot{\theta} \qquad u_3 = \dot{\psi}$$

(2) Form angular velocities and linear velocities of every body, mass center, and external force point of application:

$${}^{G}\vec{\omega}^{A} = \dot{\psi}\hat{k} = u_3\hat{k} \qquad {}^{G}\vec{V}^{A^*} = \frac{d}{2}\dot{\psi}\hat{j}$$

$${}^{G}\vec{\omega}^{B} = (\dot{\psi}+\dot{\theta})\hat{k} = (u_2+u_3)\hat{k}$$

$${}^{G}\vec{V}^{P} = {}^{G}\vec{V}^{Q} + ({}^{G}\vec{\omega}^{B} \times \vec{r}^{QP}) + {}^{B}\vec{V}^{P}$$

$$= d\dot{\psi}\hat{j} + \left[(\dot{\psi}+\dot{\theta})\hat{k} \times (L+s)(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\right]$$

$$+ (\dot{s}\cos\theta\,\hat{i} + \dot{s}\sin\theta\,\hat{j})$$

$$= \left[d\dot{\psi} + (\dot{\psi}+\dot{\theta})(L+s)\cos\theta + \dot{s}\sin\theta\right]\hat{j}$$

$$+ \left[\dot{s}\cos\theta - (\dot{\psi}+\dot{\theta})(L+s)\sin\theta\right]\hat{i}$$

Therefore

$${}^{G}\vec{\omega}^{A} = u_3\,\hat{k} \qquad {}^{G}\vec{V}^{A*} = \frac{d}{2}u_3\hat{j}$$

$${}^{G}\vec{V}^{P} = [\cos\theta\,\hat{i} + \sin\theta\,\hat{j}]u_1 + [(L+r)(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})]u_2$$
$$+ [(L+r)(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) + d\,\hat{j}]u_3$$

$${}^{G}\vec{\omega}^{B} = (u_2 + u_3)\hat{k}$$

(3) Form and tabulate all partial angular velocities and partial velocities:

$${}^{G}\vec{\omega}_1^{A} = 0 \qquad {}^{G}\vec{V}_1^{P} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad {}^{G}\vec{\omega}_1^{B} = 0$$

$${}^{G}\vec{\omega}_2^{A} = 0 \qquad {}^{G}\vec{V}_2^{P} = (L+r)(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \qquad {}^{G}\vec{\omega}_3^{B} = \hat{k}$$

$${}^{G}\vec{\omega}_3^{A} = \hat{k} \qquad {}^{G}\vec{V}_3^{P} = (L+r)(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) + d\,\hat{j} \qquad {}^{G}\vec{\omega}_2^{B} = \hat{k}$$

$${}^{G}\vec{V}_1^{A*} = 0 \qquad {}^{G}\vec{V}_2^{A*} = 0 \qquad {}^{G}\vec{V}_3^{A*} = \frac{d}{2}\hat{j}$$

(4) Form accelerations of all mass centers and angular accelerations of all bodies:

$${}^{G}\vec{\alpha}^{A} = \ddot{\psi}\,\hat{k} = \dot{u}_3\,\hat{k}$$

$${}^{G}\vec{a}^{A*} = \left[{}^{G}\vec{\omega}^{A} \times ({}^{G}\vec{\omega}^{A} \times \vec{r}^{CA*})\right] + \left({}^{G}\vec{\alpha}^{A} \times \vec{r}^{CA*}\right)$$
$$= \left[u_3\hat{k} \times (u_3\hat{k} \times \tfrac{d}{2}\hat{i})\right] + \left(\dot{u}_3\,\hat{k} \times \tfrac{d}{2}\hat{i}\right)$$
$$= -\frac{d}{2}u_3^2\,\hat{i} + \frac{d}{2}\dot{u}_3\,\hat{j}$$

$${}^G\vec{a}^P = {}^G\vec{a}^Q + \left[{}^G\vec{\omega}^B \times ({}^G\vec{\omega}^B \times \vec{r}^{QP})\right] \times ({}^G\vec{\alpha}^B \times \vec{r}^{QP})$$
$$+ {}^B\vec{a}^P + 2\,({}^G\vec{\omega}^B \times {}^B\vec{v}^P)$$

$${}^G\vec{a}^Q = \left[{}^G\vec{\omega}^A \times ({}^G\vec{\omega}^A \times \vec{r}^{CQ})\right] + \left[{}^G\vec{\alpha}^A \times \vec{r}^{CQ}\right]$$
$$= \left[u_3\hat{k} \times (u_3\hat{k} \times d\hat{\imath})\right] + (\dot{u}_3\hat{k} \times d\hat{\imath})$$
$$= -u_3^2 d\hat{\imath} + d\dot{u}_3\hat{\jmath}$$

$${}^G\vec{\omega}^B \times ({}^G\vec{\omega}^B \times \vec{r}^{QP}) = (u_2+u_3)\hat{k} \times (u_2+u_3)\hat{k} \times \left[(L+s)(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\right]$$
$$= (u_2+u_3)^2(L+s)(-\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$$

$$({}^G\vec{\alpha}^B \times \vec{r}^{QP}) = (\dot{u}_2+\dot{u}_3)\hat{k} \times (L+s)(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$
$$= (L+s)(\dot{u}_2+\dot{u}_3)(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath})$$

$${}^B\vec{a}^P = \dot{u}_1(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

$$2\,({}^G\vec{\omega}^B \times {}^B\vec{v}^P) = 2\,(u_2+u_3)\hat{k} \times u_1(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$
$$= 2\,(u_2+u_3)(u_1)(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath})$$

Therefore:

$${}^G\vec{a}^P = -u_3^2 d\hat{\imath} + d\dot{u}_3\hat{\jmath} + (u_2+u_3)^2(L+s)(-\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$$
$$+ (L+s)(\dot{u}_2+\dot{u}_3)(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath})$$
$$+ \dot{u}_1(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) + 2(u_2+u_3)(u_1)(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath})$$

Summary:

$${}^{G}\vec{\alpha}^{A} = \dot{u}_3 \hat{k}$$

$${}^{G}\vec{a}^{A*} = -u_3^2 \, d/2 \, \hat{\imath} + \dot{u}_3 \frac{d}{2} \hat{\jmath}$$

$${}^{G}\vec{a}^{P} = \Big[-u_3^2 d - (L+s)(u_2+u_3)^2\cos\theta - (L+s)(\dot{u}_2+\dot{u}_3)\sin\theta + \dot{u}_1\cos\theta$$
$$-2(u_2+u_3)u_1\sin\theta\Big]\hat{\imath}$$
$$+\Big[d\dot{u}_3 - (L+s)(u_2+u_3)^2\sin\theta + (L+s)(\dot{u}_2+\dot{u}_3)\cos\theta + \dot{u}_1\sin\theta$$
$$+2(u_2+u_3)u_1\cos\theta\Big]\hat{\jmath}$$

(5) Form inertia forces and inertia torques:

$$\vec{T}^{*} = -I_3 \dot{u}_3 \hat{k}$$

$$\vec{R}_P^{*} = -m\,{}^{G}\vec{a}^{P}$$

$$\vec{R}_A^{*} = -M\,{}^{G}\vec{a}^{A*}$$

Note:

$$\vec{T}^{*} = -\left[\alpha_1 I_1 - \omega_2\omega_3(I_2 - I_3)\right]\hat{\imath}$$
$$-\left[\alpha_2 I_2 - \omega_3\omega_1(I_3 - I_1)\right]\hat{\jmath}$$
$$-\left[\alpha_3 I_3 - \omega_1\omega_2(I_1 - I_2)\right]\hat{k}$$

(6) Form generalized active forces and generalized inertia forces:

Generalized inertia forces:

$$F_r^* = {}^G\vec{V}_r^P \cdot (-m\,{}^G\vec{a}^P) + {}^G\vec{\omega}_r^A \cdot \vec{T}^* + {}^G\vec{V}_r^{A^*} \cdot (-M\,{}^G\vec{a}^{A^*})$$

$$F_1^* = (\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \cdot (-m)({}^G\vec{a}^P) + 0 \cdot \vec{T}^* + 0 \cdot (-M)({}^G\vec{a}^{A^*})$$

$$= -m\cos\theta\left[-u_3^2 d - (L+s)(u_2+u_3)^2\cos\theta - (L+s)(\dot{u}_2+\dot{u}_3)\sin\theta + \dot{u}_1\cos\theta - 2(u_2+u_3)u_1\sin\theta\right]$$

$$-m\sin\theta\left[d\dot{u}_3 - (L+s)(u_2+u_3)^2\sin\theta + (L+s)(\dot{u}_2+\dot{u}_3)\cos\theta + \dot{u}_1\sin\theta + 2(u_2+u_3)u_1\cos\theta\right]$$

$$= -m\left[-u_3^2 d\cos\theta - (L+s)(u_2+u_3)^2 + \dot{u}_1 + d\dot{u}_3\sin\theta\right]$$

$$= m\left[u_3^2 d\cos\theta + (L+s)(u_2+u_3)^2 - \dot{u}_1 - d\dot{u}_3\sin\theta\right]$$

$$F_2^* = (L+s)(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \cdot (-m)({}^G\vec{a}^P) + 0 \cdot \vec{T}^* + 0 \cdot (-M)({}^G\vec{a}^{A*})$$

$$= m(L+s)\sin\theta\Big[-u_3^2 d - (L+s)(u_2+u_3)^2\cos\theta - (L+s)(\dot{u}_2+\dot{u}_3)\sin\theta + \dot{u}_1\cos\theta - 2(u_2+u_3)u_1\sin\theta\Big]$$

$$- m(L+s)\cos\theta\Big[d\dot{u}_3 - (L+s)(u_2+u_3)^2\sin\theta + (L+s)(\dot{u}_2+\dot{u}_3)\cos\theta + \dot{u}_1\sin\theta + 2(u_2+u_3)u_1\cos\theta\Big]$$

$$= m\Big[-(L+s)u_3^2 d\sin\theta - (L+s)d\dot{u}_3\cos\theta - (L+s)^2(\dot{u}_2+\dot{u}_3) - (L+s)2(u_2+u_3)u_1\Big]$$

$$F_3^* = \Big\{-(L+s)\sin\theta\,\hat{i} + [(L+s)\cos\theta + d]\,\hat{j}\Big\} \cdot (-m)({}^G\vec{a}^P) + \hat{k} \cdot \vec{T}^* + \left(\frac{d}{2}\hat{j}\right) \cdot (-M)({}^G\vec{a}^{A*})$$

$$f_3^* = m(L+s)\sin\theta\Big[-u_3^2 d - (L+s)(u_2+u_3)^2\cos\theta - (L+s)(\dot u_2+\dot u_3)\sin\theta + \dot u_1\cos\theta - 2(u_2+u_3)u_1\sin\theta\Big]$$

$$- m(L+s)\cos\theta\Big[d\dot u_3 - (L+s)(u_2+u_3)^2\sin\theta + (L+s)(\dot u_2+\dot u_3)\cos\theta + \dot u_1\sin\theta + 2(u_2+u_3)u_1\cos\theta\Big]$$

$$- md\Big[d\dot u_3 - (L+s)(u_2+u_3)^2\sin\theta + (L+s)(\dot u_2+\dot u_3)\cos\theta + \dot u_1\sin\theta + 2(u_2+u_3)u_1\cos\theta\Big]$$

$$- I_3\dot u_3 - M\left(\frac{d}{2}\right)^2\dot u_3$$

$$= -\left(I_3 + M\left(\frac{d}{2}\right)^2\right)\dot u_3 - m(L+s)u_3^2 d\sin\theta - m(L+s)\cos\theta\, d\dot u_3 - m(L+s)^2(\dot u_2+\dot u_3) - 2m(u_2+u_3)(L+s)u_1 - md^2\dot u_3 + md(L+s)(u_2+u_3)^2\sin\theta - md(L+s)(\dot u_2+\dot u_3)\cos\theta - md\dot u_1\sin\theta - 2md(u_2+u_3)u_1\cos\theta$$

$$= -\left(I_3 + M\left(\frac{d}{2}\right)^2\right)\dot u_3 - md^2\dot u_3 - m(\dot u_2+\dot u_3)(L+s)^2 - 2m(u_2+u_3)(L+s)u_1 - md\dot u_1\sin\theta - 2md u_1 u_2\cos\theta - 2md u_1 u_3\cos\theta - 2md(L+s)\dot u_3\cos\theta - md(L+s)\dot u_2\cos\theta + md(L+s)u_2^2\sin\theta + 2md(L+s)u_2u_3\sin\theta$$

Summary:

$$F_1^* = m\left[u_3^2 d\cos\theta + (L+s)(u_2+u_3)^2 - \dot{u}_1 - d\dot{u}_3\sin\theta\right]$$

$$F_2^* = m\Big[-(L+s)u_3^2 d\sin\theta - (L+s)d\dot{u}_3\cos\theta - (L+s)^2(\dot{u}_2+\dot{u}_3) - 2(L+s)(u_2+u_3)u_1\Big]$$

$$\begin{aligned}F_3^* = &-\left(I_3 + M\left(\tfrac{d}{2}\right)^2\right)\dot{u}_3 - md^2\dot{u}_2 - m(\dot{u}_2+\dot{u}_3)(L+s)^2 \\ &- 2m(u_2+u_3)(L+s)u_1 - md\dot{u}_1\sin\theta - 2md\,u_1u_2\cos\theta \\ &- 2md\,u_1u_3\cos\theta - 2md(L+s)\dot{u}_3\cos\theta - md(L+s)\dot{u}_2\cos\theta \\ &+ md(L+s)u_2^2\sin\theta + 2md(L+s)u_2u_3\sin\theta\end{aligned}$$

Generalized Active Forces:

$$\begin{aligned}F_r = &\,{}^G\vec{V}_r^P \cdot \left[-(K_T s + b_T u_1)(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\right] \\ &+ {}^G\vec{\omega}_r^A \cdot \left[T\hat{k}_r + K\theta\hat{k} + b_r u_2\hat{k}\right] \\ &+ {}^G\vec{\omega}_r^B \cdot \left[-K_r\theta\hat{k} - b_r u_2\hat{k}\right]\end{aligned}$$

Forces ~~that~~ don't contribute to the generalized active forces: weights, normal forces between P and B, pin reactions at C and Q.

$$f_1 = -(K_T s + b_T u_1)$$

$$f_2 = -k_r \theta - b_r u_2$$

$$f_3 = T$$

(7) Form Equations of Motion:

$$F_r + F_r^* = 0$$

$$m\left[u_3^2 d \cos\theta + (L+s)(u_2+u_3)^2 - \dot{u}_1 - d\,\dot{u}_3 \sin\theta\right] - K_T s - b_T u_1 = 0$$

$$m\left[-(L+s)u_3^2 d \sin\theta - (L+s)d\,\dot{u}_3 \cos\theta - (L+s)^2(\dot{u}_2+\dot{u}_3) - (L+s)(2)(u_2+u_3)u_1\right] - k_r\theta - b_r u_2 = 0$$

$$-\left[I_3 + M\left(\tfrac{d}{2}\right)^2\right]\dot{u}_3 - md^2\dot{u}_3 - m(\dot{u}_2+\dot{u}_3)(L+s)^2 - 2m(u_2+u_3)(L+s)u_1 - md\,\dot{u}_1\sin\theta - 2md\,u_1u_2\cos\theta - 2md\,u_1u_3\cos\theta - 2md(L+s)\dot{u}_3\cos\theta - md(L+s)\dot{u}_2\cos\theta + md(L+s)u_2^2\sin\theta + 2md(L+s)u_2u_3\sin\theta + T = 0$$

Derivation of Equations of Motion

3. Lagrange's Equations

Generalized Coordinates : ψ, s, θ

$$T = \frac{1}{2} I_c \dot{\psi}^2 + \frac{1}{2} m \, {}^G V^{P^2}$$

$$V = \frac{1}{2} K_T s^2 + \frac{1}{2} K_r \theta^2$$

$$D = \frac{1}{2} b_T \dot{s}^2 + \frac{1}{2} b_r \dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i$$

$q_1 = s$
$q_2 = \theta$
$q_3 = \psi$

$${}^G\vec{V}^P = {}^G\vec{V}^Q + \left({}^G\vec{\omega}^B \times \vec{r}\right) + {}^B\vec{V}^P$$

$$= d\dot{\psi}\hat{j} + \left[(\dot{\psi} + \dot{\theta})\hat{k} \times (L+s)(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\right]$$
$$+ (\dot{s}\cos\theta\,\hat{i} + \dot{s}\sin\theta\,\hat{j})$$

$$= d\dot{\psi}\hat{j} + (\dot{\psi} + \dot{\theta})(L+s)\cos\theta\,\hat{j} - (\dot{\psi} + \dot{\theta})(L+s)\sin\theta\,\hat{i}$$
$$+ \dot{s}\cos\theta\,\hat{i} + \dot{s}\sin\theta\,\hat{j}$$

$$= \left[d\dot{\psi} + (\dot{\psi} + \dot{\theta})(L+s)\cos\theta + \dot{s}\sin\theta\right]\hat{j}$$
$$+ \left[\dot{s}\cos\theta - (\dot{\psi} + \dot{\theta})(L+s)\sin\theta\right]\hat{i}$$

$$|{}^G\vec{V}^P|^2 = \left[d\dot{\psi} + (\dot{\psi}+\dot{\theta})(L+s)\cos\theta + \dot{s}\sin\theta\right]^2 + \left[\dot{s}\cos\theta - (\dot{\psi}+\dot{\theta})(L+s)\sin\theta\right]^2$$

This reduces after several steps to:

$$({}^G V^P)^2 = d^2\dot{\psi}^2 + \dot{s}^2 + (\dot{\psi}+\dot{\theta})^2(L+s)^2 + 2d\dot{\psi}\dot{s}\sin\theta + 2d\dot{\psi}(L+s)(\dot{\psi}+\dot{\theta})\cos\theta$$

Therefore

$$T = \frac{1}{2}I_c\dot{\psi}^2 + \frac{1}{2}m\left[d^2\dot{\psi}^2 + \dot{s}^2 + (\dot{\psi}+\dot{\theta})^2(L+s)^2 + 2d\dot{\psi}\dot{s}\sin\theta + 2d\dot{\psi}(L+s)(\dot{\psi}+\dot{\theta})\cos\theta\right]$$

$$V = \frac{1}{2}K_T s^2 + \frac{1}{2}K_r\theta^2$$

$$D = \frac{1}{2}b_T\dot{s}^2 + \frac{1}{2}b_r\dot{\theta}^2$$

$q_1 = s$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}}\right) - \frac{\partial T}{\partial s} + \frac{\partial V}{\partial s} + \frac{\partial D}{\partial \dot{s}} = Q_1$$

$$\frac{\partial T}{\partial \dot{s}} = m\dot{s} + md\dot{\psi}\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}}\right) = m\ddot{s} + md\ddot{\psi}\sin\theta + md\dot{\psi}\dot{\theta}\cos\theta$$

$$\frac{\partial T}{\partial s} = m(\dot{\psi}+\dot{\theta})^2(L+s) + md\dot{\psi}(\dot{\psi}+\dot{\theta})\cos\theta$$

$$\frac{\partial V}{\partial s} = K_T s \qquad Q_1 = 0$$

$$\frac{\partial D}{\partial \dot{s}} = b_T \dot{s}$$

Eq. of Motion:

$$m\ddot{s} + md\ddot{\psi}\sin\theta + md\dot{\psi}\dot{\theta}\cos\theta - m(\dot{\psi}+\dot{\theta})^2(L+s) - md\dot{\psi}(\dot{\psi}+\dot{\theta})\cos\theta + K_T s + b_T\dot{s} = 0$$

$$m\ddot{s} + md\ddot{\psi}\sin\theta - m(L+s)(\dot{\psi}+\dot{\theta})^2 - md\dot{\psi}^2\cos\theta + K_T s + b_T\dot{s} = 0 \qquad [1]$$

$$\underline{\underline{q_2 = \theta}}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} + \frac{\partial D}{\partial \dot\theta} = Q_2$$

$$\frac{\partial T}{\partial \dot\theta} = m(\dot\psi + \dot\theta)(L+s)^2 + md\dot\psi(L+s)\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = m(\ddot\psi + \ddot\theta)(L+s)^2 + 2m(\dot\psi + \dot\theta)(L+s)\dot s$$
$$+ md\ddot\psi(L+s)\cos\theta + md\dot\psi\dot s\cos\theta$$
$$- md\dot\psi(L+s)\dot\theta\sin\theta$$

$$\frac{\partial T}{\partial \theta} = md\dot\psi\dot s\cos\theta - md\dot\psi(L+s)(\dot\psi + \dot\theta)\sin\theta$$

$$\frac{\partial V}{\partial \theta} = K_r\theta \qquad \frac{\partial D}{\partial \dot\theta} = b_r\dot\theta \qquad Q_2 = 0$$

$$m(\ddot\psi + \ddot\theta)(L+s)^2 + 2m(\dot\psi + \dot\theta)(L+s)\dot s + md\ddot\psi(L+s)\cos\theta$$
$$+ md\dot\psi\dot s\cos\theta - md\dot\psi(L+s)\dot\theta\sin\theta - md\dot\psi\dot s\cos\theta$$
$$+ md\dot\psi(L+s)(\dot\psi + \dot\theta)\sin\theta + K_r\theta + b_r\dot\theta = 0$$

$$m(\ddot\psi + \ddot\theta)(L+s)^2 + 2m(L+s)(\dot\psi + \dot\theta)\dot s + md\ddot\psi(L+s)\cos\theta$$
$$+ md\dot\psi^2(L+s)\sin\theta + K_r\theta + b_r\dot\theta = 0$$

[2]

$q_3 = \psi$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\psi}\right) - \frac{\partial T}{\partial \psi} + \frac{\partial V}{\partial \psi} + \frac{\partial D}{\partial \dot\psi} = Q_3$$

$$\frac{\partial T}{\partial \dot\psi} = I_c \dot\psi + m d^2 \dot\psi + m(\dot\psi + \dot\theta)(L+s)^2 + m d \dot s \sin\theta + 2 m d \dot\psi (L+s)\cos\theta + m d (L+s)\dot\theta \cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\psi}\right) = I_c \ddot\psi + m d^2 \ddot\psi + m(\ddot\psi + \ddot\theta)(L+s)^2 + 2m(\dot\psi + \dot\theta)(L+s)\dot s + m d \ddot s \sin\theta + m d \dot s \dot\theta \cos\theta + 2 m d \ddot\psi (L+s)\cos\theta + 2 m d \dot\psi \dot s \cos\theta - 2 m d \dot\psi (L+s)\dot\theta \sin\theta + m d \dot s \dot\theta \cos\theta + m d (L+s)\ddot\theta \cos\theta - m d (L+s)\dot\theta^2 \sin\theta$$

$$\frac{\partial T}{\partial \psi} = 0 \qquad \frac{\partial D}{\partial \dot\psi} = 0 \qquad Q_3 = T$$

$$\frac{\partial V}{\partial \psi} = 0$$

Eq. of Motion

$$I_c \ddot\psi + m d^2 \ddot\psi + m(\ddot\psi + \ddot\theta)(L+s)^2 + 2m(\dot\psi + \dot\theta)(L+s)\dot s + m d \ddot s \sin\theta + 2 m d \dot s \dot\theta \cos\theta + 2 m d \ddot\psi (L+s)\cos\theta + 2 m d \dot\psi \dot s \cos\theta - 2 m d \dot\psi (L+s)\dot\theta \sin\theta + m d (L+s)\ddot\theta \cos\theta - m d (L+s)\dot\theta^2 \sin\theta = T$$

[3]

Summary: Equations of Motion

$$m\ddot{r} + md\ddot{\psi}\sin\theta - m(L+r)(\dot{\psi}+\dot{\theta})^2 - md\dot{\psi}^2\cos\theta + K_T r + b_T\dot{r} = 0 \qquad [1]$$

$$m(\ddot{\psi}+\ddot{\theta})(L+r)^2 + 2m(L+r)(\dot{\psi}+\dot{\theta})\dot{r} + md\ddot{\psi}(L+r)\cos\theta + md\dot{\psi}^2(L+r)\sin\theta + K_r\theta + b_r\dot{\theta} = 0 \qquad [2]$$

$$I_c\ddot{\psi} + md^2\ddot{\psi} + m(\ddot{\psi}+\ddot{\theta})(L+r)^2 + 2m(\dot{\psi}+\dot{\theta})(L+r)\dot{r} + md\ddot{r}\sin\theta + 2md\dot{r}\dot{\theta}\cos\theta + 2md\ddot{\psi}(L+r)\cos\theta + 2md\dot{\psi}\dot{r}\cos\theta - 2md\dot{\psi}(L+r)\dot{\theta}\sin\theta + md(L+r)\ddot{\theta}\cos\theta - md(L+r)\dot{\theta}^2\sin\theta = T \qquad [3]$$