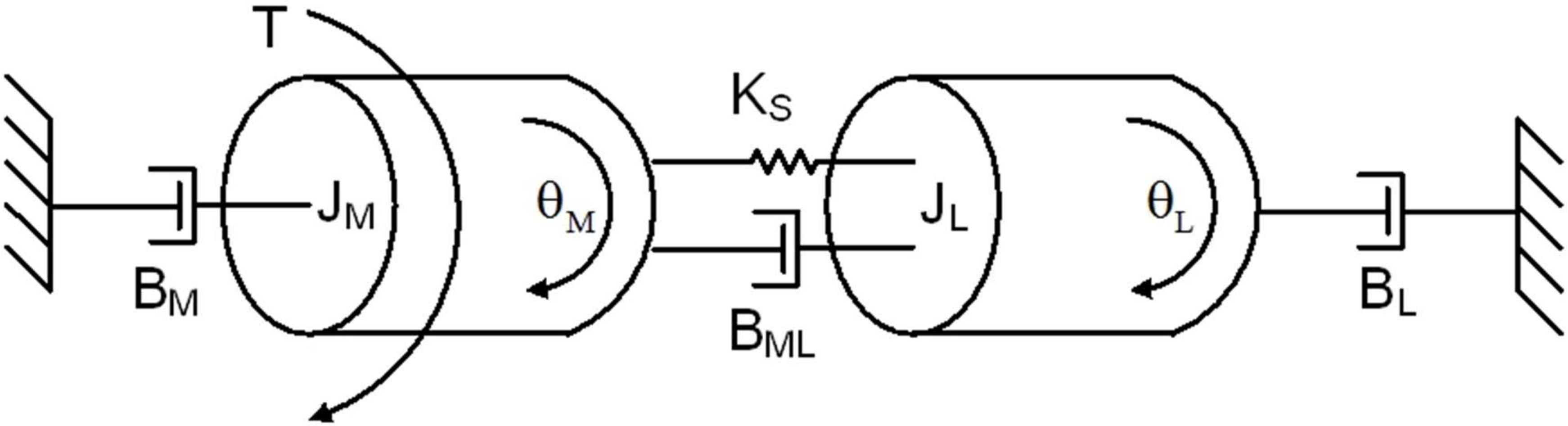
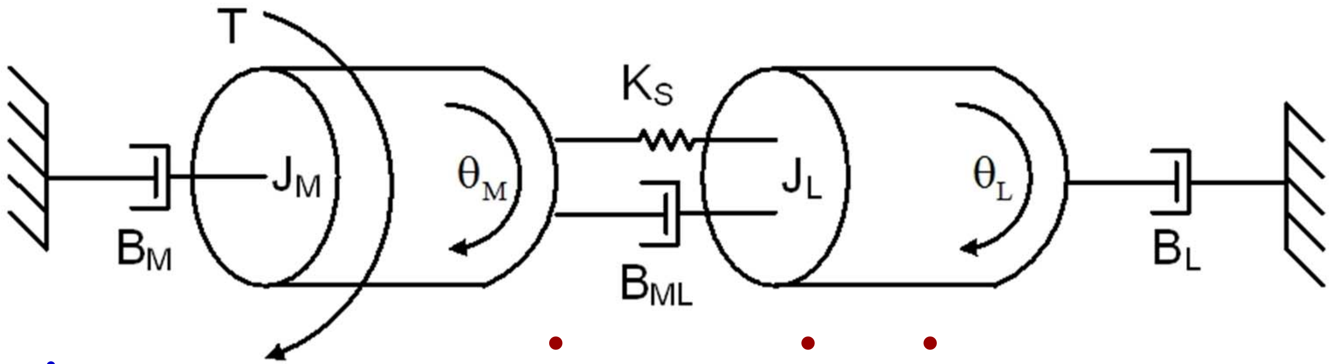
Modeling of Compliance



J_M = rotor inertia of a motor
J_L = driven-load inertia
K_S = elasticity of coupling
B_{ML} = viscous damping of coupling
B_M = viscous damping between ground and motor rotor
B_L = viscous damping between ground and load inertia
T = electromagnetic torque applied to motor rotor

- Comments
  - K_S, the elasticity of the coupling; it is often neglected in low-power systems; modeling it in high-power systems is essential.
  - B_{ML}, the viscous damping of the coupling; it is usually small, as transmission materials provide little damping.
  - B_M and B_L can be neglected in the following analysis, as they have a small effect on resonance. They are included here for completeness.
  - Coulomb friction has been neglected. The fixed value of Coulomb friction has little impact on stability when the motor is moving. At rest, the impact of stiction on resonance is more complex. Sometimes stiction is thought of as increasing the load inertia when the motor is at rest. This accounts for the tendency of systems to change resonance behavior when the motion stops.

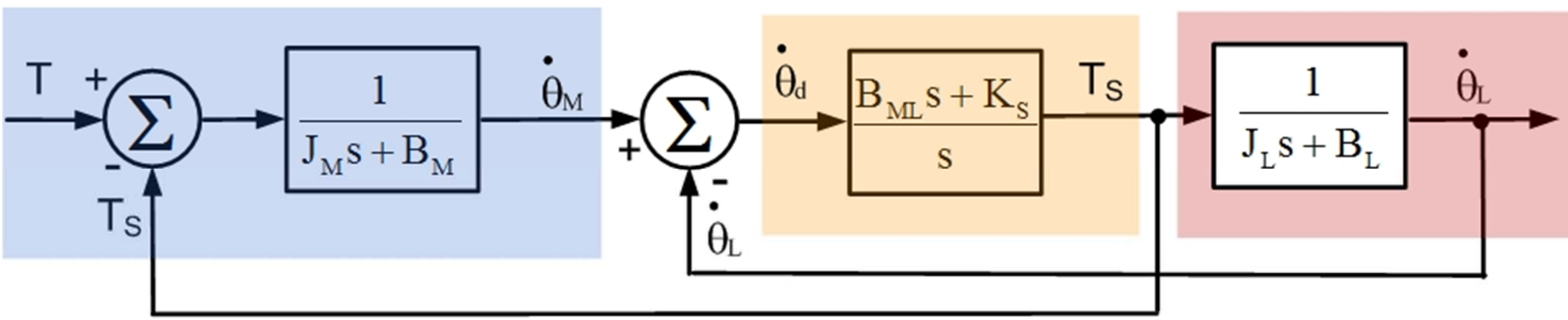


Equations of Motion

$$T - B_M \dot{\theta}_M - B_{ML}(\dot{\theta}_M - \dot{\theta}_L) - K_S(\theta_M - \theta_L) = J_M \ddot{\theta}_M$$

$$-B_L \dot{\theta}_L + B_{ML}(\dot{\theta}_M - \dot{\theta}_L) + K_S(\theta_M - \theta_L) = J_L \ddot{\theta}_L$$

$$\dot{\theta}_d = \dot{\theta}_M - \dot{\theta}_L$$



$$J_M \ddot{\theta}_M + B_M \dot{\theta}_M = T - T_S$$

$$T_S = B_{ML} \dot{\theta}_d + K_S \theta_d$$

$$J_L \ddot{\theta}_L + B_L \dot{\theta}_L = T_S$$

Equations of Motion

$$T - B_M \dot{\theta}_M - B_{ML}(\dot{\theta}_M - \dot{\theta}_L) - K_S(\theta_M - \theta_L) = J_M \ddot{\theta}_M$$

$$-B_L \dot{\theta}_L + B_{ML}(\dot{\theta}_M - \dot{\theta}_L) + K_S(\theta_M - \theta_L) = J_L \ddot{\theta}_L$$

Laplace Transform of the Equations of Motion

$$\left[J_M s^2 + (B_{ML} + B_M)s + K_S\right]\Theta_M(s) = (B_{ML}s + K_S)\Theta_L(s) + T(s)$$

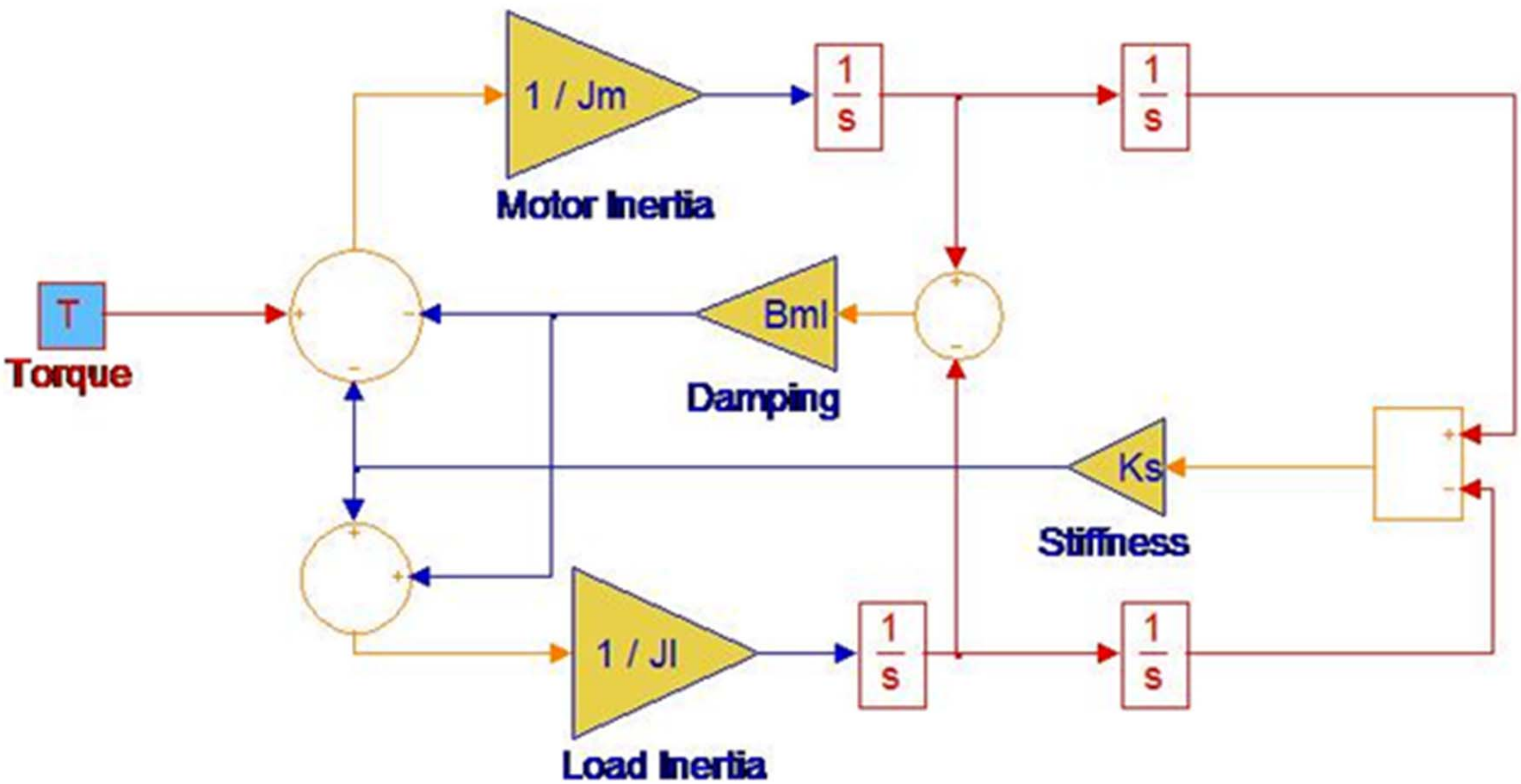
$$\left[J_L s^2 + (B_{ML} + B_L)s + K_S\right]\Theta_L(s) = (B_{ML}s + K_S)\Theta_M(s)$$

$$\begin{bmatrix} J_M s^2 + (B_{ML} + B_M)s + K_S & -(B_{ML}s + K_S) \\ -(B_{ML}s + K_S) & J_L s^2 + (B_{ML} + B_L)s + K_S \end{bmatrix} \begin{bmatrix} \Theta_M(s) \\ \Theta_L(s) \end{bmatrix} = \begin{bmatrix} T(s) \\ 0 \end{bmatrix}$$

MatLab / Simulink Block Diagram (B_M = 0 and B_L = 0)

$$T - B_{ML}(\dot{\theta}_M - \dot{\theta}_L) - K_S(\theta_M - \theta_L) = J_M \ddot{\theta}_M$$

$$B_{ML}(\dot{\theta}_M - \dot{\theta}_L) + K_S(\theta_M - \theta_L) = J_L \ddot{\theta}_L$$



Transfer Functions

$$\frac{\Theta_M}{T}(s) = \frac{J_L s^2 + (B_{ML} + B_L)s + K_S}{D(s)}$$

$$\frac{\Theta_L}{T}(s) = \frac{B_{ML}s + K_S}{D(s)}$$

$$\begin{aligned} D(s) = {} & [J_M J_L]s^4 + [(J_M + J_L)B_{ML} + J_M B_L + J_L B_M]s^3 + \\ & [(J_M + J_L)K_S + B_M B_L + B_{ML}(B_L + B_M)]s^2 + [(B_M + B_L)K_S]s \end{aligned}$$

Transfer Functions

(B_L = 0 and B_M = 0)

$$\frac{\Theta_M}{T}(s)=\left[\frac{1}{(J_M+J_L)s^2}\right]\left[\frac{J_Ls^2+B_{ML}s+K_S}{\frac{J_LJ_M}{J_L+J_M}s^2+B_{ML}s+K_S}\right]$$

$$\frac{\Theta_L}{T}(s)=\left[\frac{1}{(J_M+J_L)s^2}\right]\left[\frac{B_{ML}s+K_S}{\frac{J_LJ_M}{J_L+J_M}s^2+B_{ML}s+K_S}\right]$$

As $K_S\to\infty$

or as $s\to 0$

$$\Theta_M(s)=\Theta_L(s)=\frac{1}{(J_M+J_L)s^2}\quad \text{Rigid-Body Motion}$$

Transfer Functions in Standard Form
(B_M = 0 and B_L = 0)

$$\frac{\Theta_M}{T}(s)=\frac{K\left[\frac{s^2}{\omega_{AR}^2}+\frac{2\zeta_{AR}s}{\omega_{AR}}+1\right]}{s^2\left[\frac{s^2}{\omega_R^2}+\frac{2\zeta_R s}{\omega_R}+1\right]}$$

$$\frac{\Theta_L}{T}(s)=\frac{K(\tau s+1)}{s^2\left[\frac{s^2}{\omega_R^2}+\frac{2\zeta_R s}{\omega_R}+1\right]}$$

$$K=\frac{1}{J_M+J_L}$$

$$\tau=\frac{B_{ML}}{K_S}$$

$$\omega_R=\sqrt{\frac{K_S(J_M+J_L)}{J_M J_L}}$$

$$\zeta_R=\frac{B_{ML}}{2\sqrt{\frac{K_S J_M J_L}{J_M+J_L}}}$$

$$\omega_{AR}=\sqrt{\frac{K_S}{J_L}}$$

$$\zeta_{AR}=\frac{B_{ML}}{2\sqrt{K_S J_L}}$$

Natural frequency of load connected to ground through the compliance

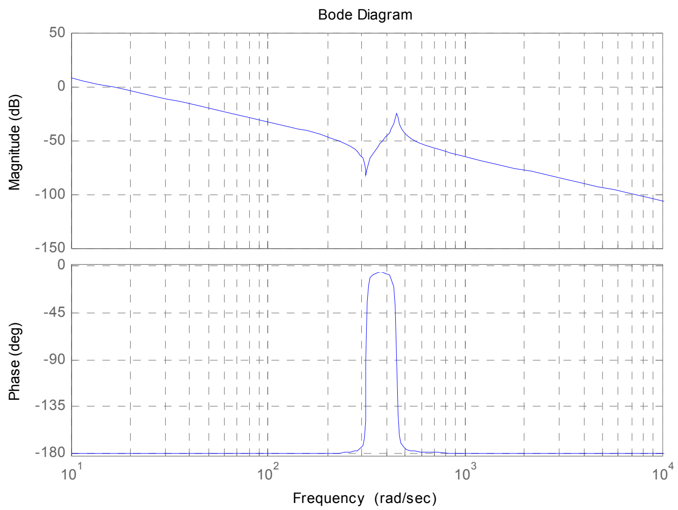
Sample Values:

$$J_L = 0.002 \text{ kg-m}^2$$

$$J_M = 0.002 \text{ kg-m}^2$$

$$K_S = 200 \text{ N-m/rad}$$

$$B_{ML} = 0.01 \text{ N-m-s/rad}$$



$$\omega_{AR} = 316 \text{ rad/s} = 50.3 \text{ Hz}$$

$$\omega_R = 447 \text{ rad/s} = 71.2 \text{ Hz}$$

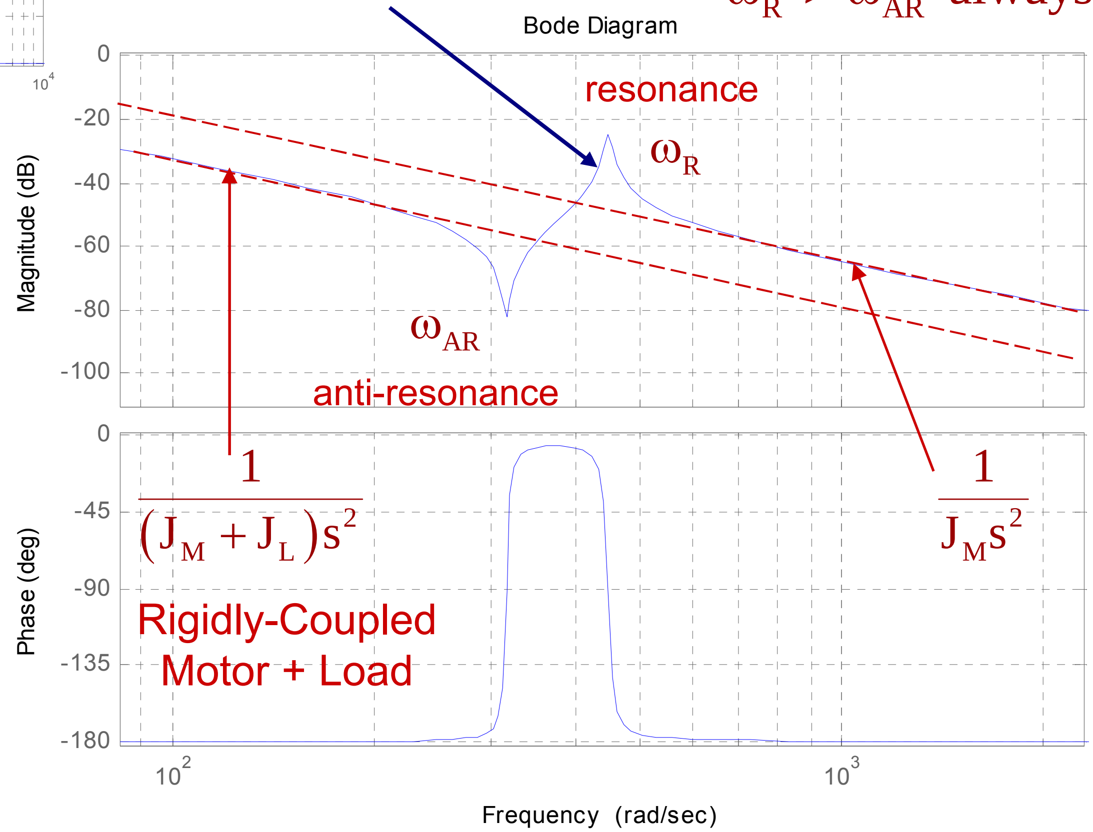
$$\zeta_{AR} = 0.008$$

$$\zeta_R = 0.011$$

$$\frac{\Theta_M}{T}(s)$$

Compliantly-Coupled Motor + Load

$\omega_R > \omega_{AR}$ always

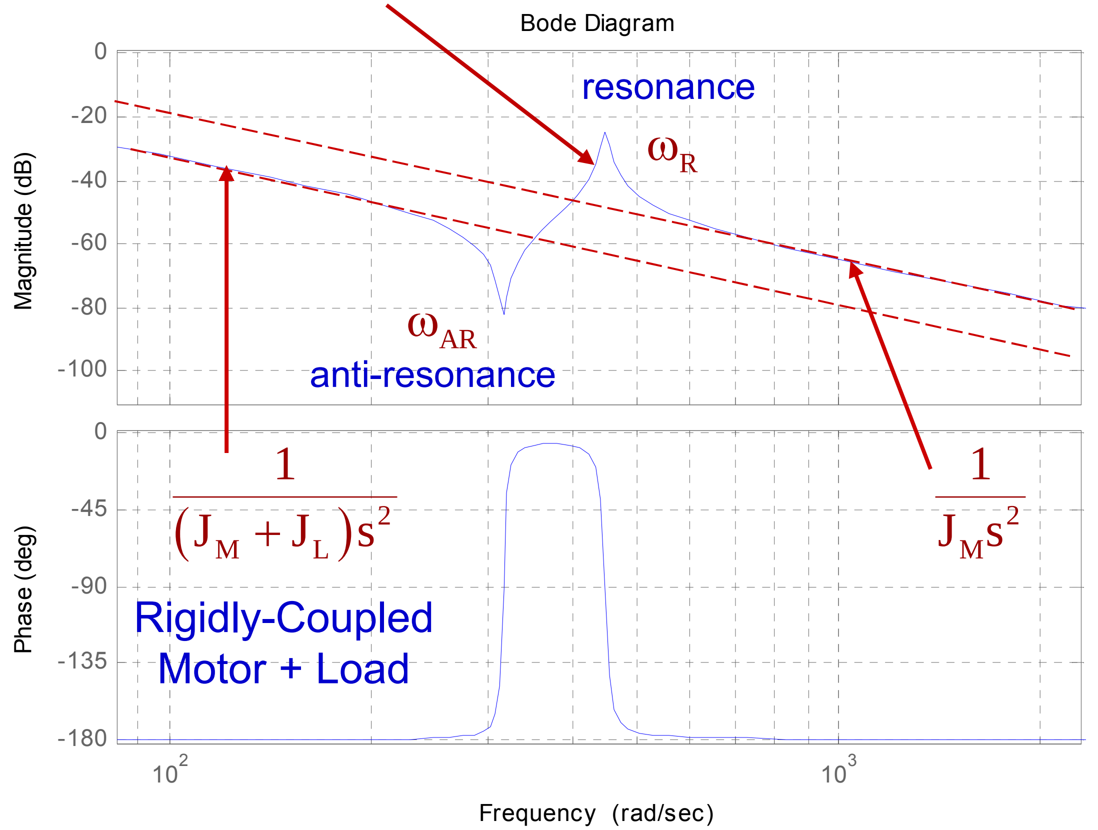


$$\frac{\Theta_M}{T}(s) = \left[\frac{1}{(J_M + J_L)s^2}\right]\left[\frac{J_L s^2 + B_{ML} s + K_S}{\frac{J_L J_M}{J_L + J_M} s^2 + B_{ML} s + K_S}\right]$$

Collocated System

$\omega_R > \omega_{AR}$ always

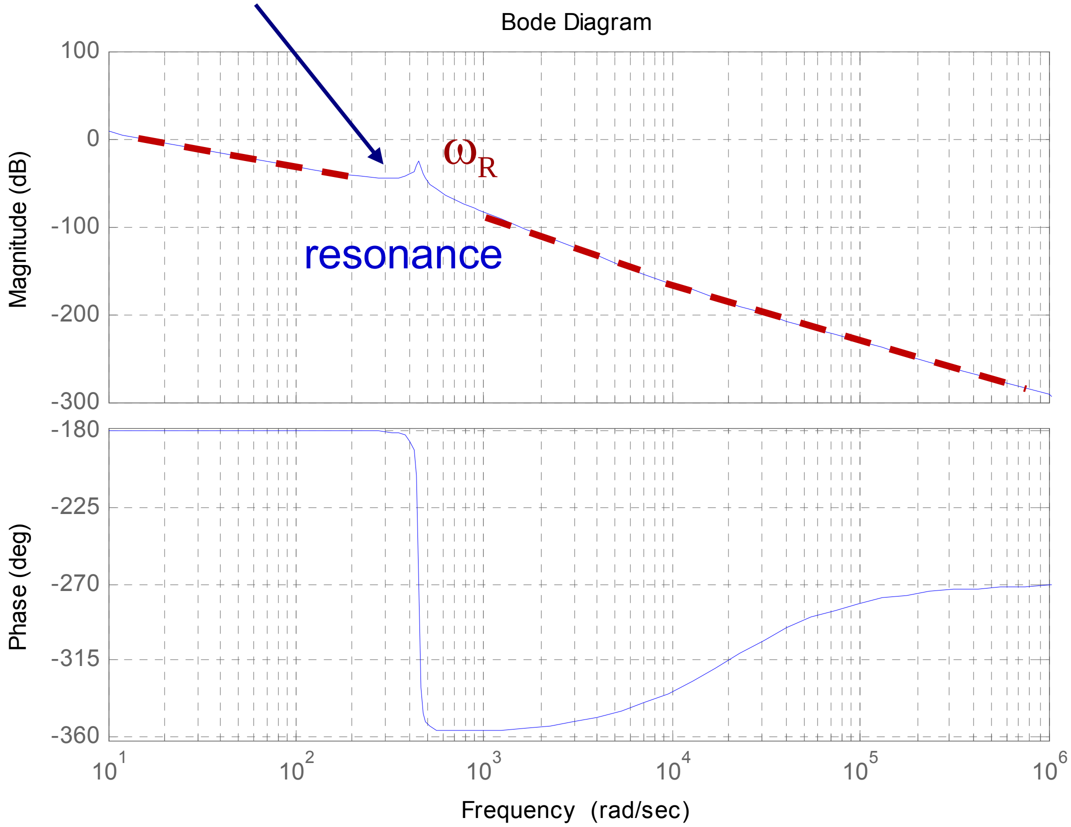
$$\frac{\Theta_M}{T}(s)$$



$$\frac{\Theta_L}{T}(s) = \left[\frac{1}{(J_M + J_L)s^2}\right]\left[\frac{B_{ML}s + K_S}{\frac{J_L J_M}{J_L + J_M}s^2 + B_{ML}s + K_S}\right]$$

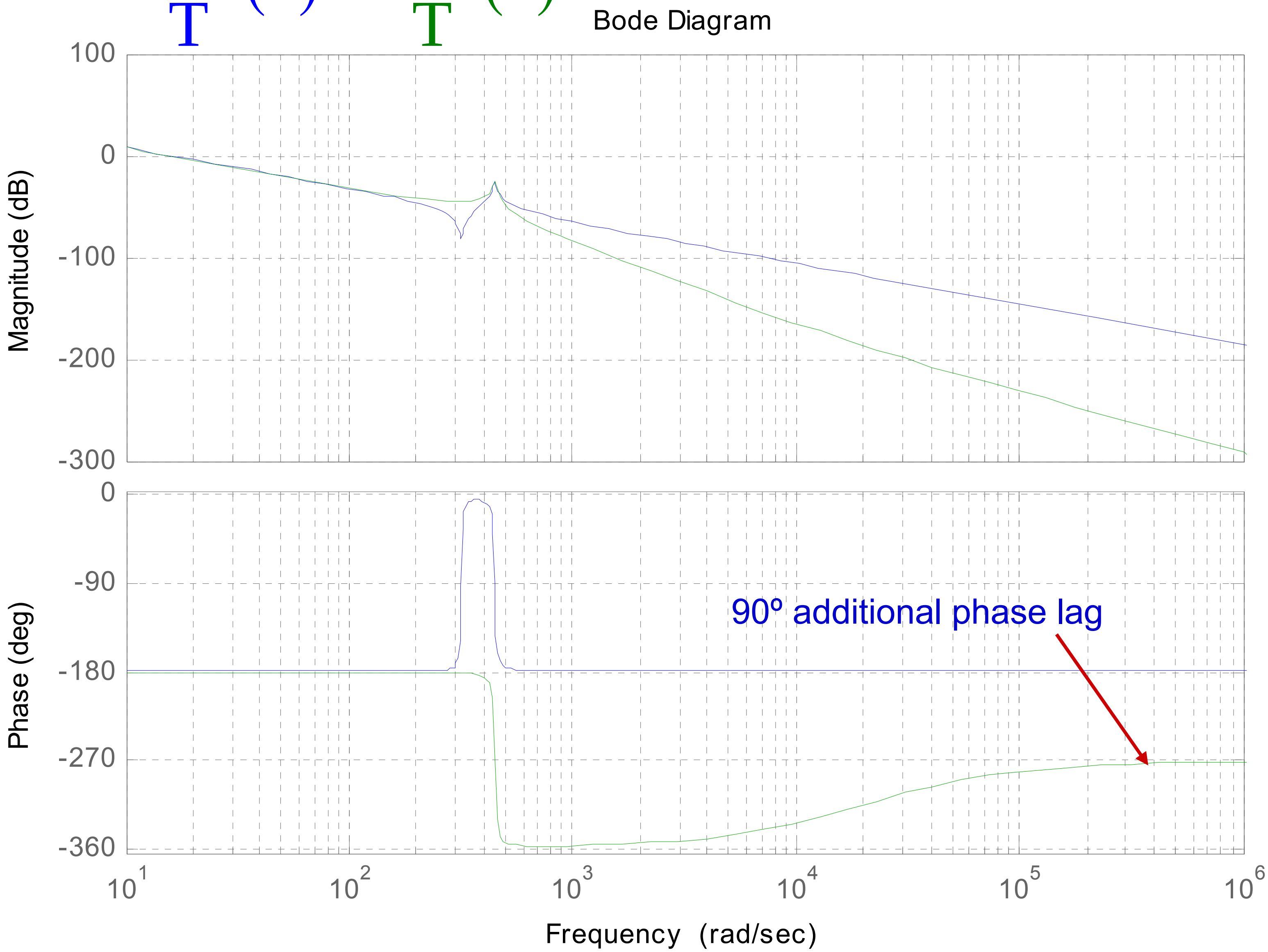
Non-Collocated System

$$\frac{\Theta_L}{T}(s)$$



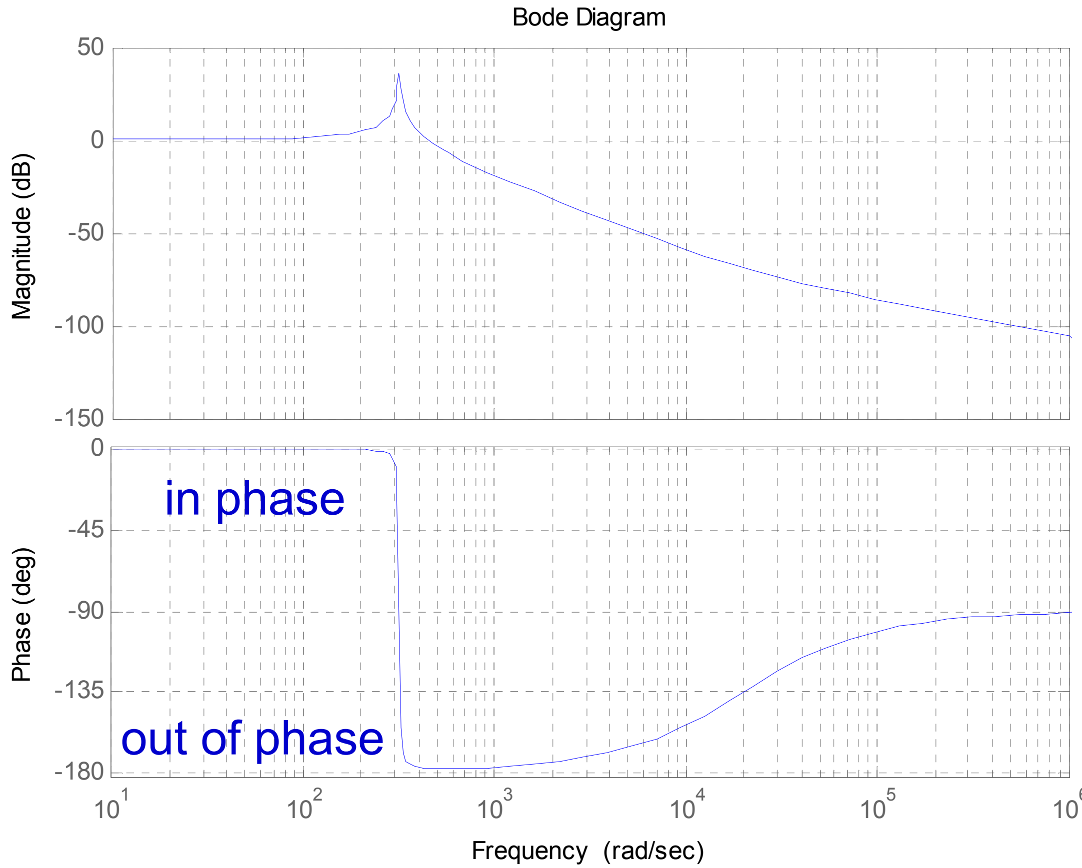
Bode Plot Comparison

$\frac{\Theta_M}{T}(s)$ $\frac{\Theta_L}{T}(s)$



$$\frac{\Theta_L}{\Theta_M}(s) = \frac{(\tau s + 1)}{\left[\frac{s^2}{\omega_{AR}^2} + \frac{2\zeta_{AR} s}{\omega_{AR}} + 1\right]}$$

$$\frac{\Theta_L}{\Theta_M}(s)$$



Note: There is no anti-resonant frequency in this transfer function. Also, there is 90° more phase lag at high frequency.

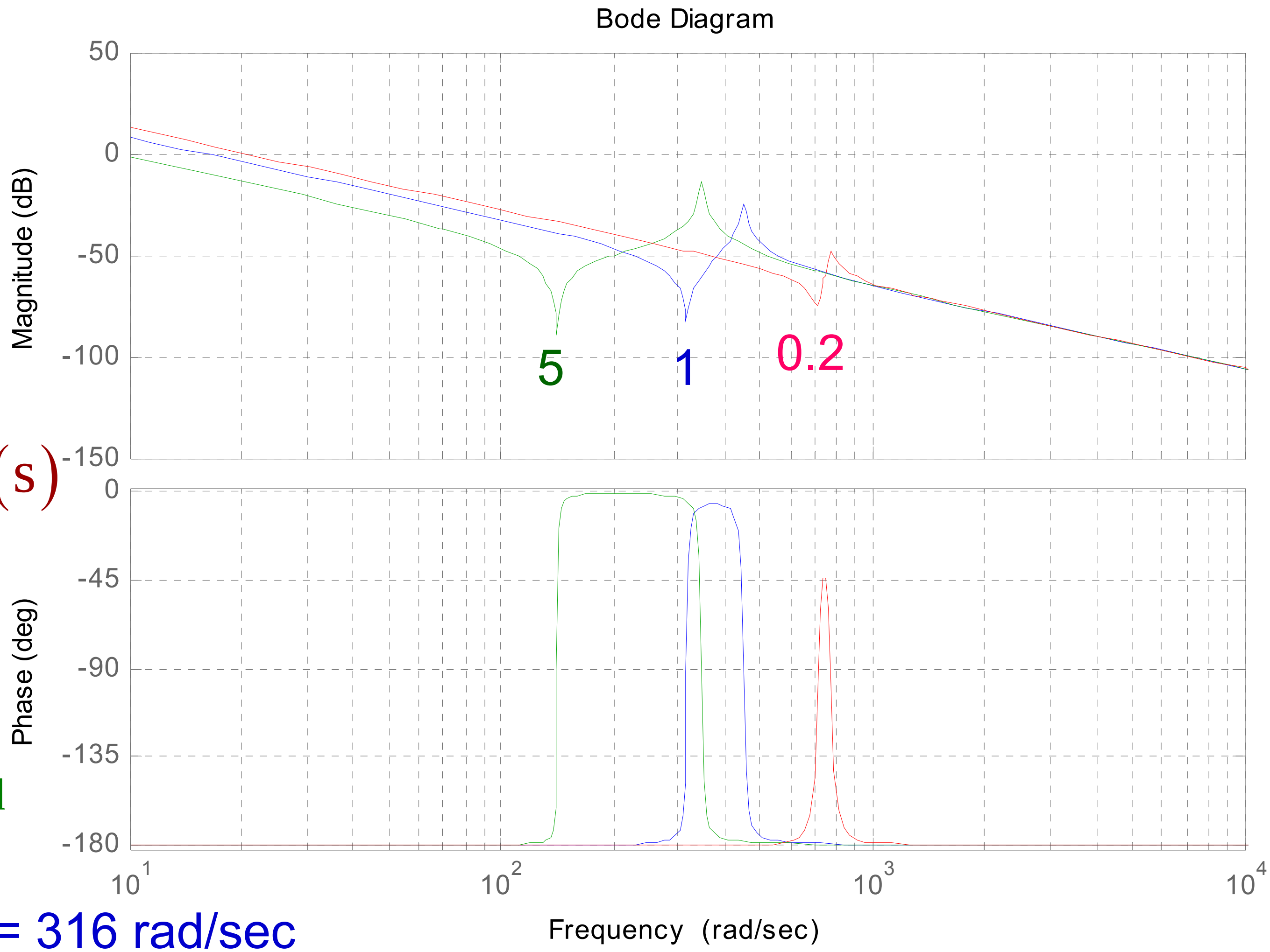
Effect of J_L / J_M Ratio on Resonance and Anti-Resonance

$$\frac{J_L}{J_M} = 1$$

$$\frac{J_L}{J_M} = 5$$

$$\frac{J_L}{J_M} = \frac{1}{5}$$

$$\frac{\Theta_M}{T}(s)$$



$$J_M = 0.002 \text{ kg-m}^2$$

$$K_S = 200 \text{ N-m/rad}$$

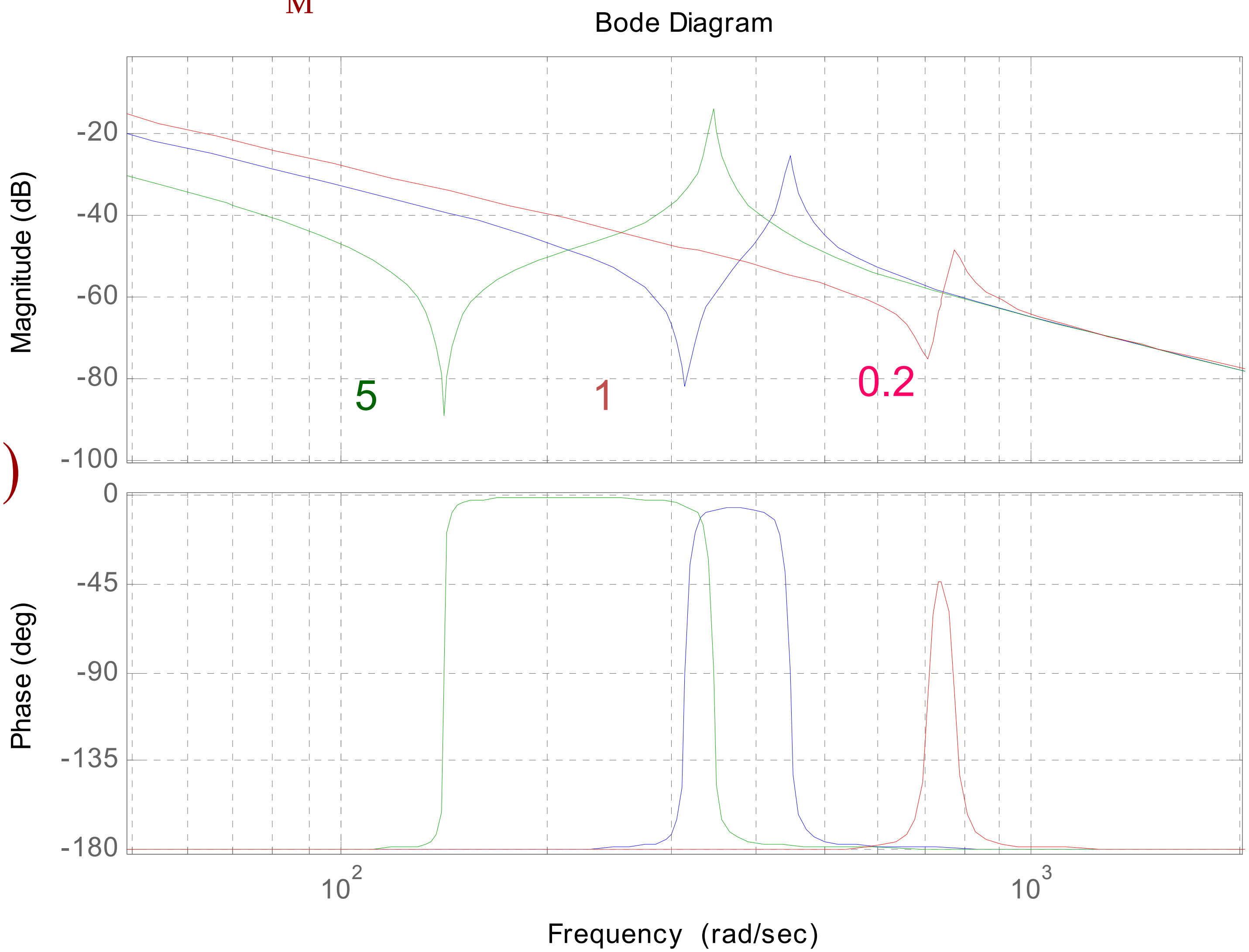
$$B_{ML} = 0.01 \text{ N-m-s/rad}$$

ω_R lower limit = 316 rad/sec

ω_{AR} lower limit = 0

Effect of Varying $\frac{J_L}{J_M}$

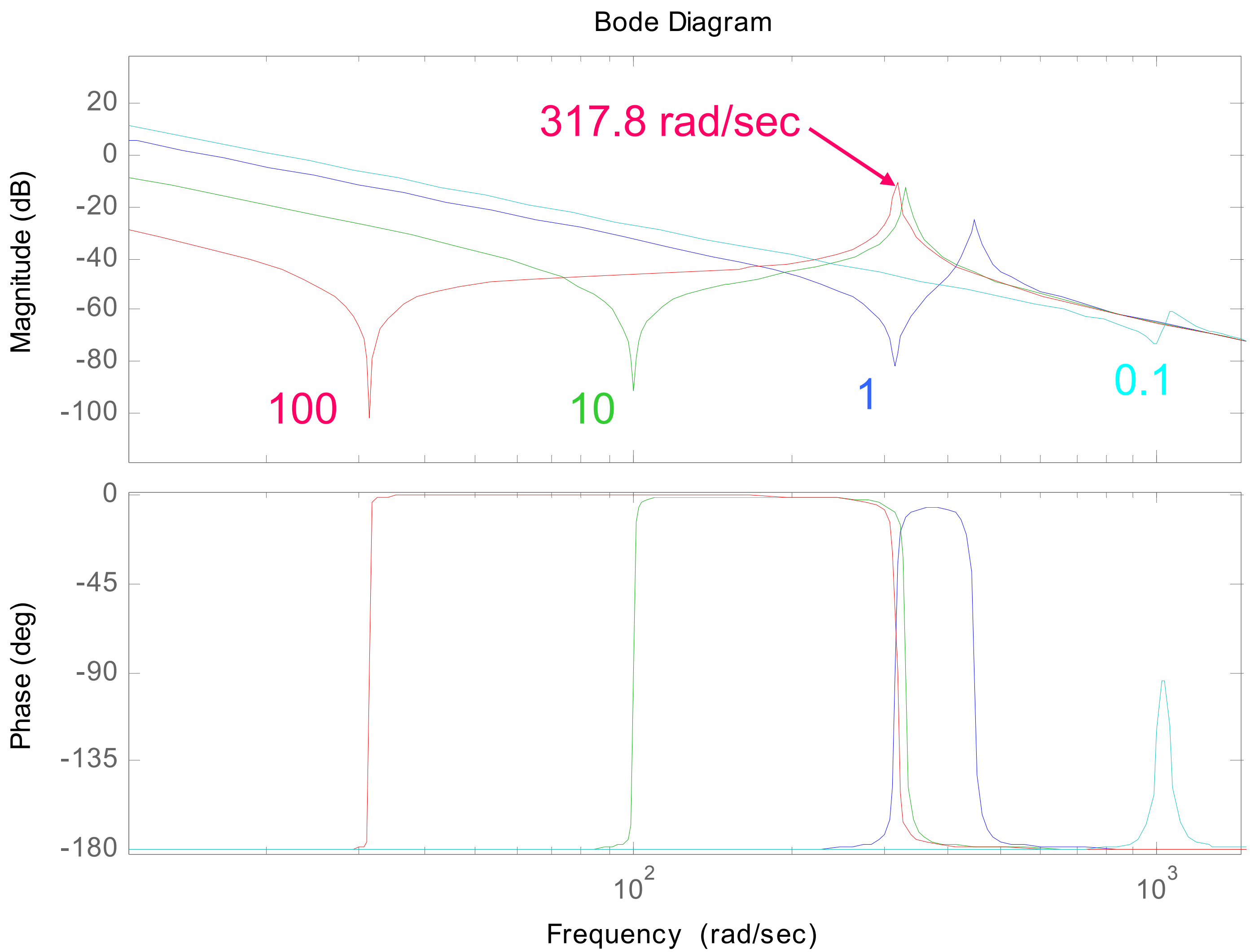
$$\frac{\Theta_M}{T}(s)$$



$$\frac{J_L}{J_M} = 100, \ 10, \ 1, \ 0.1$$

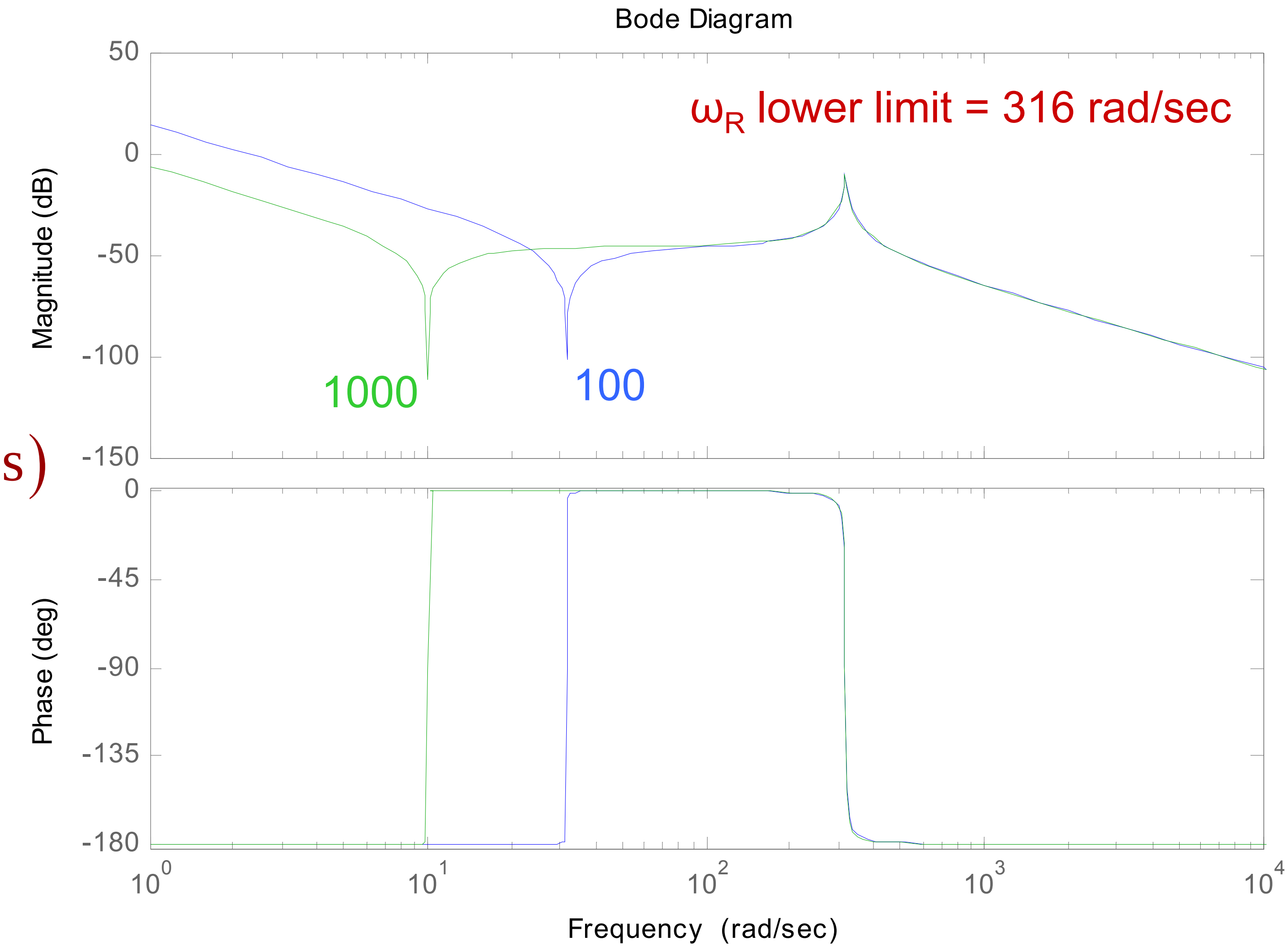
ω_R lower limit = 316 rad/sec

$$\frac{\Theta_M}{T}(s)$$



$$\frac{J_L}{J_M} = 100,\ 1000$$

$$\frac{\Theta_M}{T}(s)$$



Effect of K_S on Resonance and Anti-Resonance

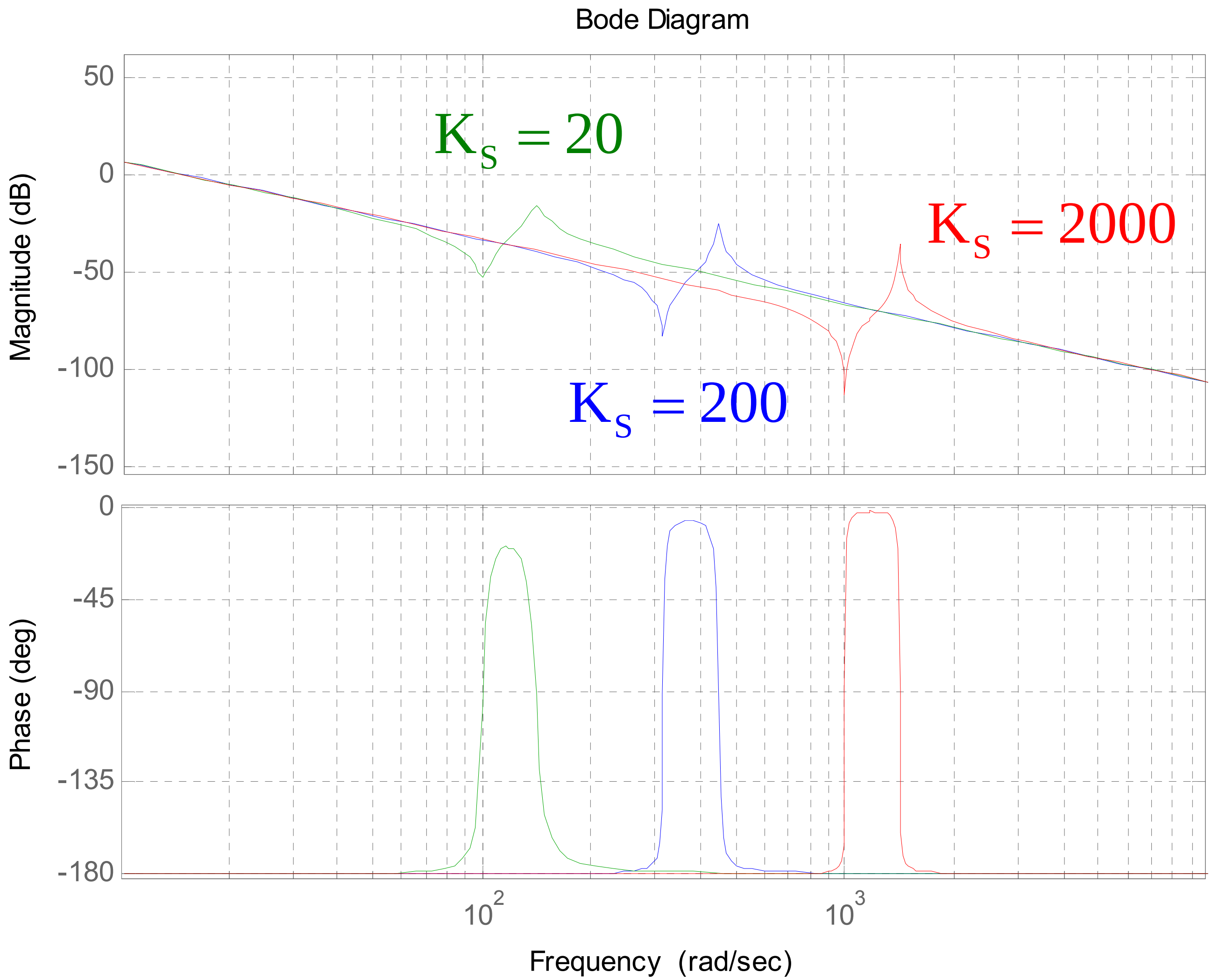
$$\frac{J_L}{J_M} = 1$$

$$K_S = 20 \ \frac{\text{N-m}}{\text{rad}}$$

$$K_S = 200 \ \frac{\text{N-m}}{\text{rad}}$$

$$K_S = 2000 \ \frac{\text{N-m}}{\text{rad}}$$

$$\frac{\Theta_M}{T}(s)$$



Limiting Behavior:

$$\frac{\Theta_M}{T}(s) = \left[\frac{1}{(J_M + J_L)s^2}\right]\left[\frac{J_L s^2 + B_{ML} s + K_S}{\frac{J_L J_M}{J_L + J_M} s^2 + B_{ML} s + K_S}\right]$$

$$\approx \frac{1}{(J_M + J_L)s^2} \quad \text{as } \omega \to 0$$

$$\approx \frac{1}{J_M s^2} \quad \text{as } \omega \to \infty$$

$$\left.\begin{array}{l} \omega_R = \sqrt{\dfrac{K_S (J_M + J_L)}{J_M J_L}} \\ \omega_{AR} = \sqrt{\dfrac{K_S}{J_L}} \end{array}\right\} \text{As } J_L \to \infty, \quad \omega_{AR} \to 0 \text{ and } \omega_R \to \sqrt{\frac{K_S}{J_M}}$$

- Observations
  - For $J_M > 0$, the anti-resonance frequency always occurs before the resonance frequency.

$$\omega_R = \sqrt{\frac{K_S\left(J_M + J_L\right)}{J_M J_L}}$$

$$\omega_{AR} = \sqrt{\frac{K_S}{J_L}}$$

  - At a low J_L/J_M ratio, the resonance and anti-resonance frequencies are close to each other at a high frequency.

- As J_L/J_M increases, both the anti-resonance and resonance frequency decrease, with the anti-resonance frequency decreasing at a faster rate.

$$\omega_R = \sqrt{\frac{K_S\left(J_M + J_L\right)}{J_M J_L}} = \sqrt{K_S \frac{1+\frac{J_L}{J_M}}{J_L}} = \sqrt{K_S \frac{1+\frac{J_M}{J_L}}{J_M}}$$

$$\omega_{AR} = \sqrt{\frac{K_S}{J_L}} = \sqrt{K_S \frac{1}{J_L}}$$

- At $J_L = J_M$, $\omega_{AR} = 0.707\omega_R$.

$$\text{As } J_L \to \infty, \quad \omega_{AR} \to 0 \text{ and } \omega_R \to \sqrt{\frac{K_S}{J_M}}$$

– For a given J_L, to increase the resonance frequency, either increase the shaft stiffness or decrease the motor inertia.

$$\omega_R = \sqrt{\frac{K_S\left(J_M + J_L\right)}{J_M J_L}} = \sqrt{K_S \frac{1+\frac{J_L}{J_M}}{J_L}} = \sqrt{K_S \frac{1+\frac{J_M}{J_L}}{J_M}}$$

– As K_S increases, both ω_R and ω_{AR} increase.

– A general guideline to avoid instability problems is to keep the desired closed-loop bandwidth well below the resonance frequency and the ratio J_L / J_M less than 5.

– Heavy-Load Approximation: $J_L > 5J_M$

$$J_M + J_L \approx J_L$$

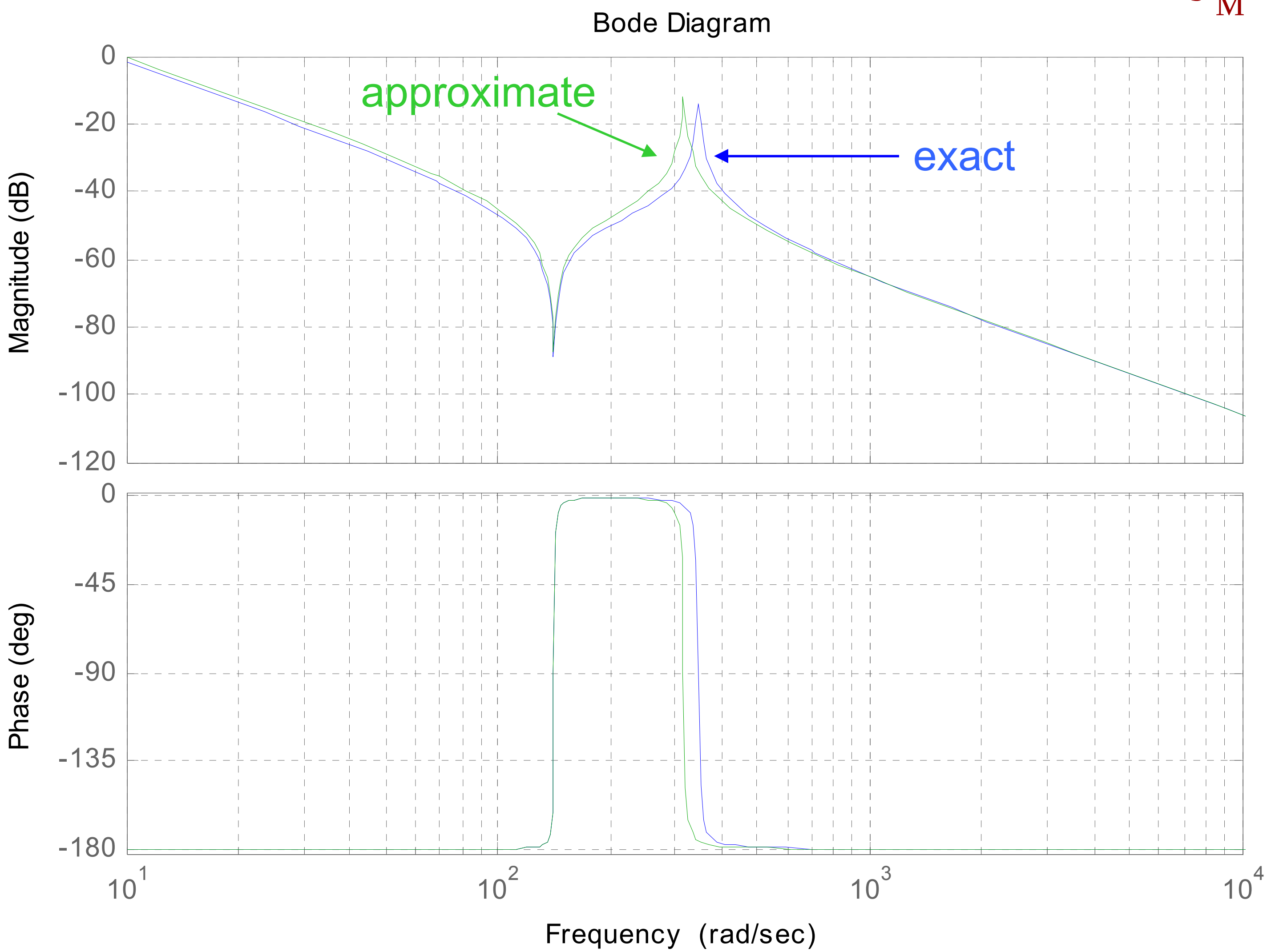
$$\frac{J_M J_L}{J_M + J_L} = \frac{J_M}{\dfrac{J_M}{J_L} + 1} \approx J_M$$

$$\frac{\Theta_M}{T}(s) = \left[\frac{1}{(J_M + J_L)s^2}\right]\left[\frac{J_L s^2 + B_{ML}s + K_S}{\dfrac{J_L J_M}{J_L + J_M}s^2 + B_{ML}s + K_S}\right]$$

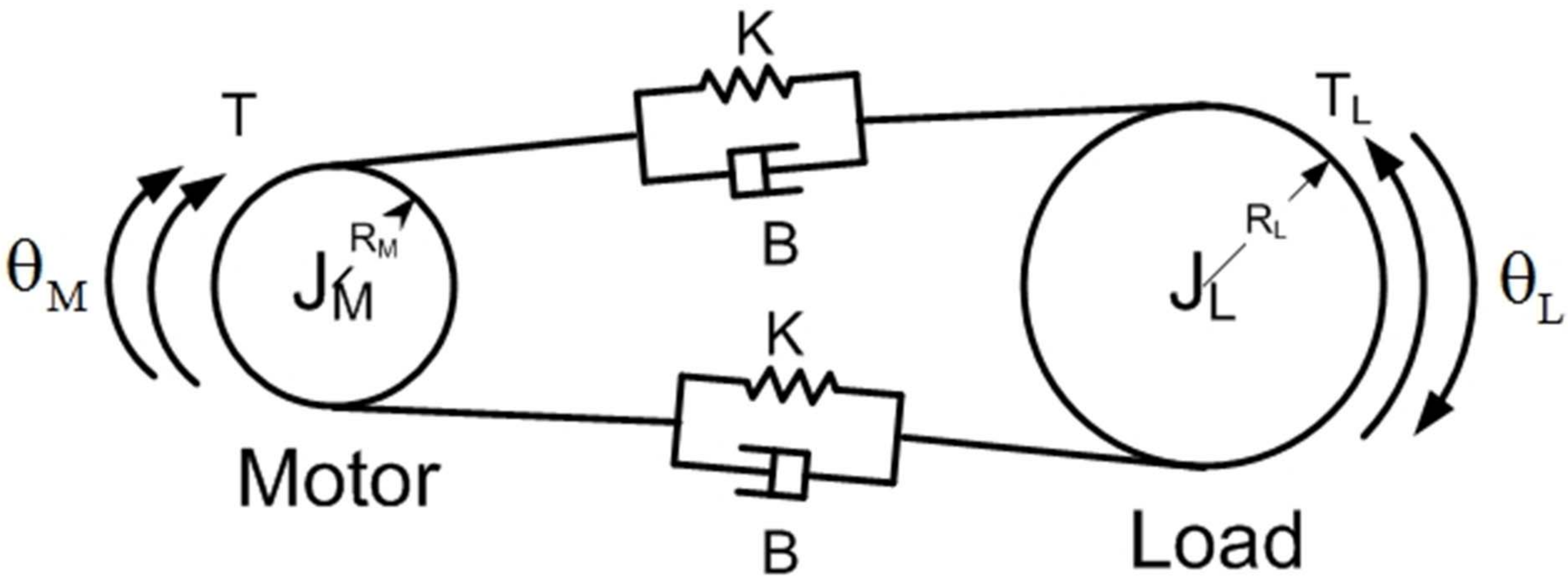
$$\approx \frac{1}{J_L s^2}\left[\frac{J_L s^2 + B_{ML}s + K_S}{J_M s^2 + B_{ML}s + K_S}\right]$$

Heavy-Load Approximation

$$\frac{J_L}{J_M} = 5$$



Belt-Driven Load and Motor

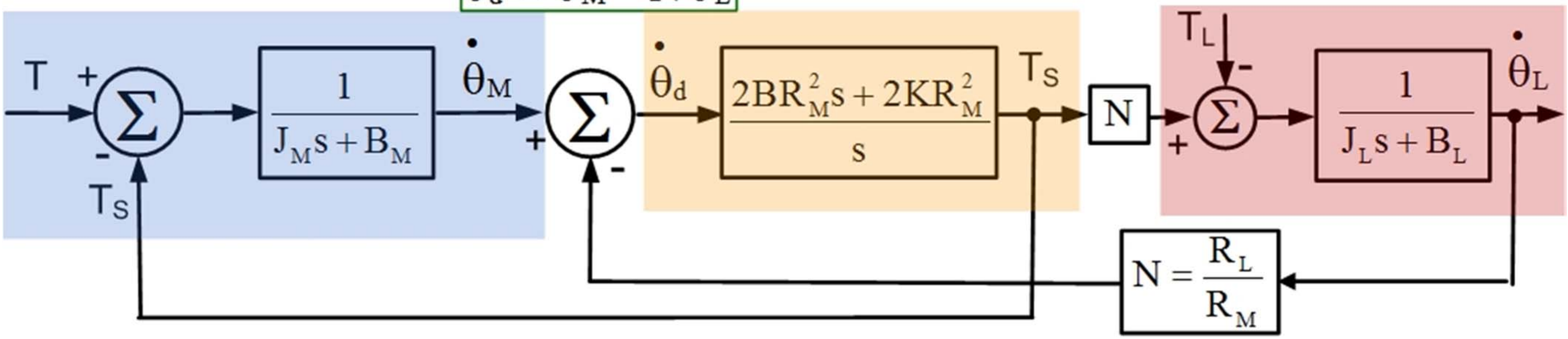


$$R_M \theta_M = R_L \theta_L$$

Rigid Belt Case:

$$\left[J_M + \left(\frac{R_M}{R_L} \right)^2 J_L \right] \ddot{\theta}_M = T$$

$$\dot{\theta}_d = \dot{\theta}_M - N\dot{\theta}_L$$



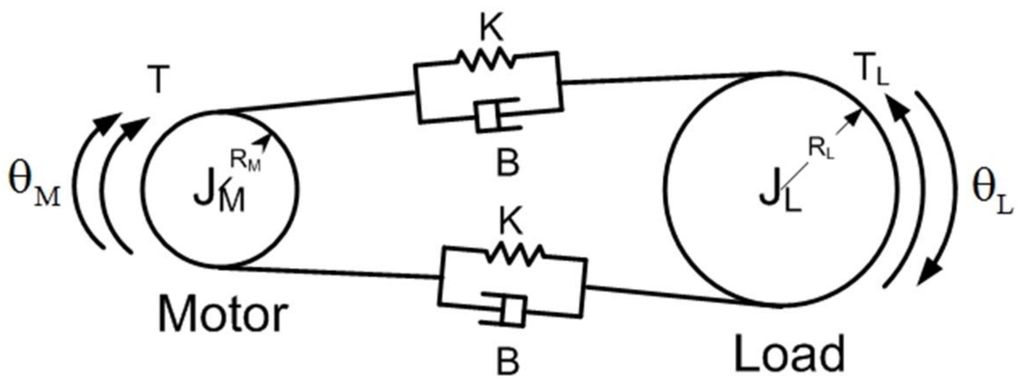
$$J_M \ddot{\theta}_M + B_M \dot{\theta}_M = T - T_S$$

$$T_S = 2BR_M^2 \dot{\theta}_d + 2KR_M^2 \theta_d$$

$$J_L \ddot{\theta}_L + B_L \dot{\theta}_L = NT_S - T_L$$

$$J_M \ddot{\theta}_M + B_M \dot{\theta}_M + 2KR_M^2 \theta_M + 2BR_M^2 \dot{\theta}_M = T + 2KR_M R_L \theta_L + 2BR_M R_L \dot{\theta}_L$$

$$J_L \ddot{\theta}_L + B_L \dot{\theta}_L + 2KR_L^2 \theta_L + 2BR_L^2 \dot{\theta}_L = 2KR_M R_L \theta_M + 2BR_M R_L \dot{\theta}_M - T_L$$



Equations of Motion

$$J_M \ddot{\theta}_M + B_M \dot{\theta}_M + 2KR_M^2\theta_M + 2BR_M^2\dot{\theta}_M = T + 2KR_M R_L \theta_L + 2BR_M R_L \dot{\theta}_L$$

$$J_L \ddot{\theta}_L + B_L \dot{\theta}_L + 2KR_L^2\theta_L + 2BR_L^2\dot{\theta}_L = 2KR_M R_L \theta_M + 2BR_M R_L \dot{\theta}_M - T_L$$

Laplace Transform of the Equations of Motion ($T_L = 0$)

$$\left[J_M s^2 + \left(B_M + 2BR_M^2\right)s + 2KR_M^2\right]\Theta_M(s) = \left(2BR_M R_L s + 2KR_M R_L\right)\Theta_L(s) + T(s)$$

$$\left[J_L s^2 + \left(B_L + 2BR_L^2\right)s + 2KR_L^2\right]\Theta_L(s) = \left(2BR_M R_L s + 2KR_M R_L\right)\Theta_M(s)$$

$$\begin{bmatrix} J_M s^2 + \left(B_M + 2BR_M^2\right)s + 2KR_M^2 & -\left(2BR_M R_L s + 2KR_M R_L\right) \\ -\left(2BR_M R_L s + 2KR_M R_L\right) & J_L s^2 + \left(B_L + 2BR_L^2\right)s + 2KR_L^2 \end{bmatrix} \begin{bmatrix} \Theta_M(s) \\ \Theta_L(s) \end{bmatrix} = \begin{bmatrix} T(s) \\ 0 \end{bmatrix}$$

Transfer Functions

$$\frac{\Theta_M}{T}(s) = \frac{J_L s^2 + \left(B_L + 2BR_L^2\right)s + 2KR_L^2}{D(s)}$$

$$\frac{\Theta_L}{T}(s) = \frac{2BR_M R_L s + 2KR_M R_L}{D(s)}$$

$$\begin{aligned} D(s) = {} & \left[J_M J_L\right]s^4 + \left[2B\left(J_L R_M^2 + J_M R_L^2\right) + J_M B_L + J_L B_M\right]s^3 + \\ & \left[2K\left(J_L R_M^2 + J_M RL\right) + B_M B_L + 2B\left(B_L R_M^2 + B_M R_L^2\right)\right]s^2 \\ & + \left[2K\left(B_L R_M^2 + B_M R_L^2\right)\right]s \end{aligned}$$

Transfer Functions
(B_L = 0 and B_M = 0)

$$\frac{\Theta_M}{T}(s) = \frac{J_L s^2 + 2BR_L^2 s + 2KR_L^2}{\left[J_M J_L\right]s^4 + \left[2B\left(J_L R_M^2 + J_M R_L^2\right)\right]s^3 + \left[2K\left(J_L R_M^2 + J_M R_L^2\right)\right]s^2}$$

$$\frac{\Theta_L}{T}(s) = \frac{2BR_M R_L s + 2KR_M R_L}{\left[J_M J_L\right]s^4 + \left[2B\left(J_L R_M^2 + J_M R_L^2\right)\right]s^3 + \left[2K\left(J_L R_M^2 + J_M R_L^2\right)\right]s^2}$$

Transfer Functions
(BL = 0 and BM = 0)

$$\frac{\Theta_M}{T}(s) = \frac{1}{\left(J_L R_M^2 + J_M R_L^2\right)s^2} \frac{J_L s^2 + 2BR_L^2 s + 2KR_L^2}{\left[\dfrac{J_M J_L}{J_L R_M^2 + J_M R_L^2} s^2 + 2Bs + 2K\right]}$$

$$\frac{\Theta_L}{T}(s) = \frac{1}{\left(J_L R_M^2 + J_M R_L^2\right)s^2} \frac{2BR_M R_L s + 2KR_M R_L}{\left[\dfrac{J_M J_L}{J_L R_M^2 + J_M R_L^2} s^2 + 2Bs + 2K\right]}$$

Transfer Functions in Standard Form

($B_M = 0$ and $B_L = 0$)

$$\frac{\Theta_M}{T}(s) = \frac{K_1\left[\dfrac{s^2}{\omega_{AR}^2} + \dfrac{2\zeta_{AR}s}{\omega_{AR}} + 1\right]}{s^2\left[\dfrac{s^2}{\omega_R^2} + \dfrac{2\zeta_R s}{\omega_R} + 1\right]}$$

$$\frac{\Theta_L}{T}(s) = \frac{K_2(\tau s + 1)}{s^2\left[\dfrac{s^2}{\omega_R^2} + \dfrac{2\zeta_R s}{\omega_R} + 1\right]}$$

$$K_2 = \frac{R_M R_L}{J_M R_M^2 + J_L R_L^2}$$

$$K_1 = \frac{R_L^2}{J_M R_L^2 + J_L R_M^2} \qquad \tau = \frac{B}{K}$$

$$\omega_R = \sqrt{\frac{2K\left(J_M R_L^2 + J_L R_M^2\right)}{J_M J_L}}$$

$$\zeta_R = \frac{B}{\sqrt{\dfrac{2KJ_M J_L}{J_M R_L^2 + J_L R_M^2}}}$$

$$\omega_{AR} = \sqrt{\frac{2KR_L^2}{J_L}}$$

$$\zeta_{AR} = \frac{BR_L}{\sqrt{2KJ_L}}$$