Novel Inverted Pendulum : Modeling + Control

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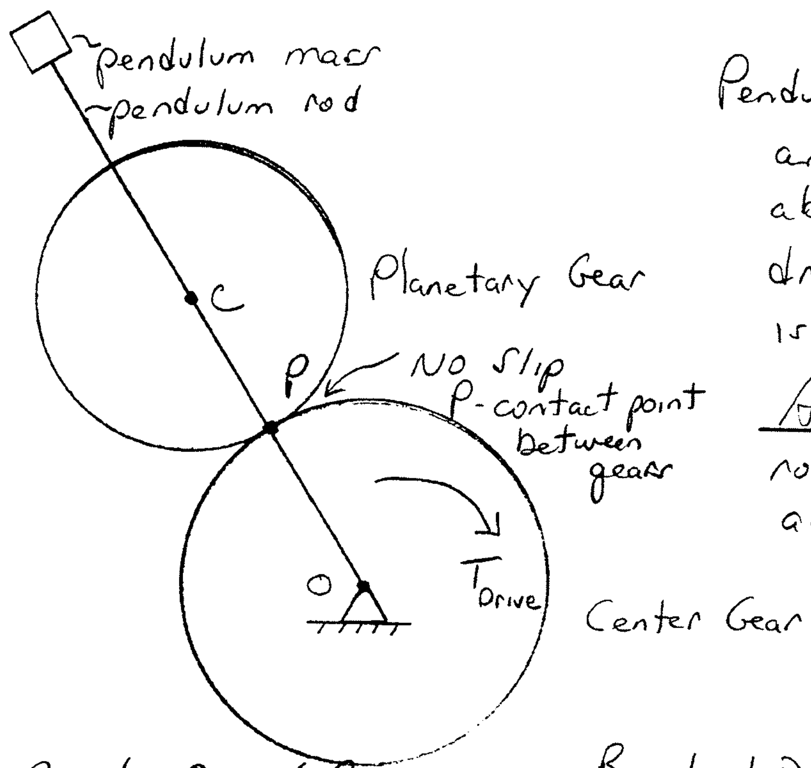
Planetary-Train Type Inverted Pendulum



Points O and C are centers of mass of the 2 gears.

Brushed DC Motor Drives Center Gear

Pendulum Arm is connected to O and C. It is free to rotate about point O, i.e., it is not driven. Only the center gear is driven.

Advantage: Unrestricted rotation angles of the motor and pendulum

Reference Frames

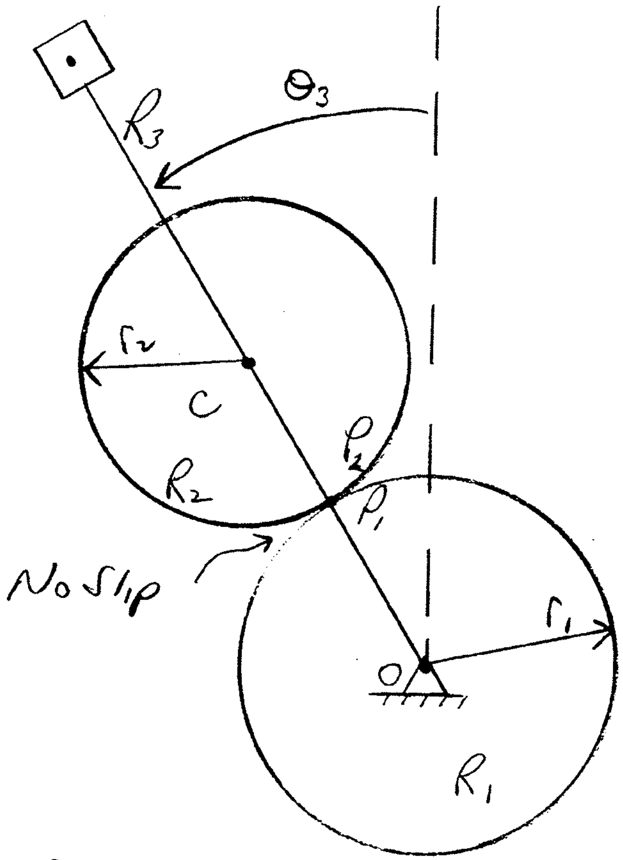
R ground

R_1 center gear

R_2 planetary gear

R_3 pendulum rod

Kinematic Analysis



ℓ = pendulum rod length

R ground

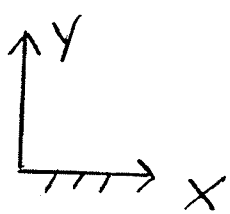
No Slip Condition: ${}^{R}\vec{V}^{P_2} = {}^{R}\vec{V}^{P_1}$

$$\left\{\begin{array}{l} P_1 \text{ is fixed on } R_1 \\ P_2 \text{ is fixed on } R_2 \\ P_1 \text{ and } P_2 \text{ are in contact at instant shown} \end{array}\right.$$

Planetary Gear Train Analysis

$$\frac{\omega_{FA}}{\omega_{LA}} = \frac{\omega_F - \omega_A}{\omega_L - \omega_A}$$

$$\frac{\omega_{13}}{\omega_{23}} = -\frac{r_2}{r_1} = \frac{\omega_{10} - \omega_{30}}{\omega_{20} - \omega_{30}}$$



$$\omega_{20} - \omega_{30} = (\omega_{10} - \omega_{30})(-r_1/r_2)$$

$$\omega_{20} = -\frac{r_1}{r_2}\omega_{10} + \omega_{30} + \frac{r_1}{r_2}\omega_{30}$$

$$\omega_{20} = -\frac{r_1}{r_2}\omega_{10} + \left(\frac{r_1+r_2}{r_2}\right)\omega_{30}$$

✓ $${}^{R}\omega^{R_2} = -\frac{r_1}{r_2}\,{}^{R}\omega^{R_1} + \left(\frac{r_1+r_2}{r_2}\right){}^{R}\omega^{R_3}$$

$$ {}^{R}\vec{V}^{P_1} = {}^{R}\vec{V}^{O} + \left({}^{R}\vec{\omega}^{R_1} \times \vec{r}_{OP_1}\right) + {}^{R_1}\vec{V}^{P_1} $$

$$ = O + \left({}^{R}\vec{\omega}^{R_1} \times \vec{r}_{OP_1}\right) + O $$

$$ = \left({}^{R}\vec{\omega}^{R_1} \times \vec{r}_{OP_1}\right) $$

$\left\{\begin{array}{l} O \text{ is fixed to ground} \\ P_1 \text{ is fixed in } R_1 \end{array}\right.$

$$ {}^{R}\vec{V}^{P_2} = {}^{R}\vec{V}^{C} + \left({}^{R}\vec{\omega}^{R_2} \times \vec{r}_{CP_2}\right) + {}^{R_2}\vec{V}^{P_2} $$

${}^{R_2}\vec{V}^{P_2} = 0$

$$ {}^{R}\vec{V}^{C} = {}^{R}\vec{V}^{O} + \left({}^{R}\vec{\omega}^{R_3} \times \vec{r}_{OC}\right) + {}^{R_3}\vec{V}^{C} $$

$\left\{\begin{array}{l} {}^{R_3}\vec{V}^{C} = 0 \\ {}^{R}\vec{V}^{O} = 0 \end{array}\right.$

$$ {}^{R}\vec{V}^{P_2} = \left({}^{R}\vec{\omega}^{R_3} \times \vec{r}_{OC}\right) + \left({}^{R}\vec{\omega}^{R_2} \times \vec{r}_{CP_2}\right) $$

$$ {}^{R}\vec{V}^{P_1} = {}^{R}\vec{V}^{P_2} $$

$$ \left({}^{R}\vec{\omega}^{R_1} \times \vec{r}_{OP_1}\right) = \left({}^{R}\vec{\omega}^{R_3} \times \vec{r}_{OC}\right) + \left({}^{R}\vec{\omega}^{R_2} \times \vec{r}_{CP_2}\right) $$

$$ \left[{}^{R}\omega^{R_1}\hat{k} \times \left(-r_1 \sin\theta_3 \hat{i} + r_1 \cos\theta_3 \hat{j}\right)\right] = \left[{}^{R}\omega^{R_3}\hat{k} \times \left(-(r_1 + r_2)\sin\theta_3 \hat{i} + (r_1+r_2)\cos\theta_3 \hat{j}\right)\right] + \left[{}^{R}\omega^{R_2}\hat{k} \times \left(r_2 \sin\theta_3 \hat{i} - r_2 \cos\theta_3 \hat{j}\right)\right] $$

$$\left(-{}^{R}\omega^{R_1} r_1 \sin\theta_3\right)\hat{j} + \left(-{}^{R}\omega^{R_1} r_1 \cos\theta_3\right)\hat{i}$$

$$= \left(-{}^{R}\omega^{R_3}\right)(r_1+r_2)\sin\theta_3\,\hat{j} + \left(-{}^{R}\omega^{R_3}\right)(r_1+r_2)\cos\theta_3\,\hat{i} + \left({}^{R}\omega^{R_2} r_2 \sin\theta_3\right)\hat{j} + \left({}^{R}\omega^{R_2} r_2 \cos\theta_3\right)\hat{i}$$

Therefore equating components:

$$\begin{cases} -{}^{R}\omega^{R_1} r_1 \cos\theta_3 = -{}^{R}\omega^{R_3}(r_1+r_2)\cos\theta_3 + {}^{R}\omega^{R_2} r_2 \cos\theta_3 \\ -{}^{R}\omega^{R_1} r_1 \sin\theta_3 = -{}^{R}\omega^{R_3}(r_1+r_2)\sin\theta_3 + {}^{R}\omega^{R_2} r_2 \sin\theta_3 \end{cases}$$

$$-{}^{R}\omega^{R_1} r_1 = -{}^{R}\omega^{R_3}(r_1+r_2) + {}^{R}\omega^{R_2} r_2$$

$${}^{R}\omega^{R_2} = -{}^{R}\omega^{R_1}\frac{r_1}{r_2} + {}^{R}\omega^{R_3}\left(\frac{r_1+r_2}{r_2}\right)$$

Does this make sense?

<u>Note:</u>

CCW is pos.

$$\begin{cases} {}^{R}\omega^{R_1} = \dot{\theta}_1 \\ {}^{R}\omega^{R_3} = \dot{\theta}_3 \end{cases}$$

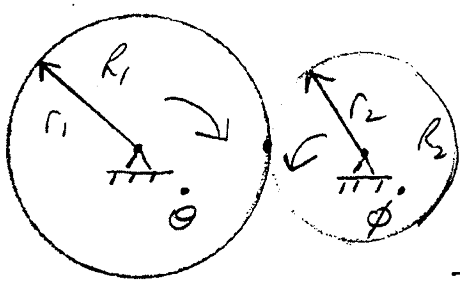
Hold R_1 fixed, i.e., ${}^{R}\omega^{R_1} = 0$.

Then $\quad {}^{R}\omega^{R_2} = \left(\dfrac{r_1 + r_2}{r_2}\right) {}^{R}\omega^{R_3}$

Now hold R_3 fixed:

$${}^{R}\omega^{R_2} = -\,{}^{R}\omega^{R_1}\,\frac{r_1}{r_2}$$

$$\frac{{}^{R}\omega^{R_2}}{{}^{R}\omega^{R_1}} = -\frac{r_1}{r_2}$$



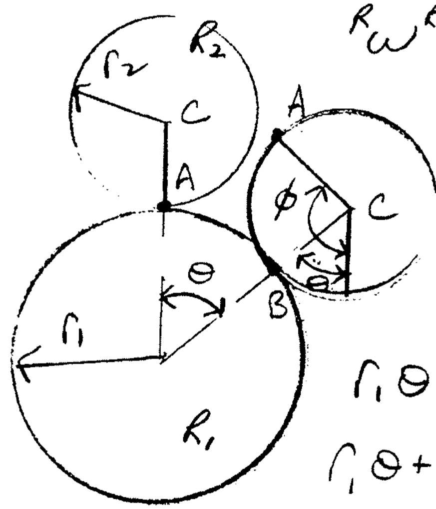
Case of 2 gear on fixed centers:

$r_1\dot{\theta} = r_2\dot{\phi}$

$$\frac{\dot{\theta}}{\dot{\phi}} = \frac{r_2}{r_1}$$

Case of a cylinder rolling without slip on another cylinder.

${}^{R}\omega^{R_2} = \dot{\phi}$



$$r_1\theta = r_2(\phi - \theta)$$

$$r_1\theta + r_2\theta = r_2\phi$$

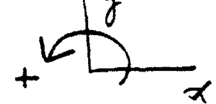
$$\phi = \frac{r_1 + r_2}{r_2}\,\theta$$

$$\dot{\phi} = \frac{r_1 + r_2}{r_2}\,\dot{\theta}$$

So $\quad {}^R\omega^{R_2} = -\,{}^R\omega^{R_1}\,\dfrac{r_1}{r_2} + {}^R\omega^{R_3}\left(\dfrac{r_1+r_2}{r_2}\right)$ ✓

This is a <u>Two-Degree-of-Freedom System</u>

positive is ccw



Use Lagrange's Equation to derive the equations of motion for the system.

Choose as generalized coordinates θ_1 and θ_3.

Equations of Motion

Kinetic Energy $T = T_1 + T_2 + T_3$

T_1 = Kinetic Energy of Center Gear $= \frac{1}{2} \bar{I}_1 \left({}^{R}\omega^{R_1}\right)^2$
(fixed-axis rotation)
$(\bar{I}_1 = I_{1_0})$

T_2 = Kinetic Energy of Planetary Gear –
(general plane motion)

$$= \frac{1}{2} m_2 \left[(r_1 + r_2)\left({}^{R}\omega^{R_3}\right)\right]^2 + \frac{1}{2} \bar{I}_2 \left({}^{R}\omega^{R_2}\right)^2$$

T_3 = Kinetic Energy of Pendulum Rod + Mass

$$= \frac{1}{2} I_{3_0} \left({}^{R}\omega^{R_3}\right)^2 + \frac{1}{2} m_p \left(l_p \, {}^{R}\omega^{R_3}\right)^2$$

Treat pendulum mass as a particle

m_p = pendulum attached mass
l_p = distance from O to pendulum mass

Potential Energy

$$V = -m_2 g\left[(r_1+r_2) - (r_1+r_2)\cos\theta_3\right] - m_p g\left[\ell_p - \ell_p\cos\theta_3\right] - m_3 g\left(\frac{\ell}{2} - \frac{\ell}{2}\cos\theta_3\right)$$

$$V = \left[-m_2 g(r_1+r_2) - m_p g \ell_p - m_3 g \frac{\ell}{2}\right](1-\cos\theta_3)$$

Generalized Coordinates $\theta_1 \quad \theta_3$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

$q_1 = \theta_1$
$q_2 = \theta_3$

Lagrange's Equations

Generalized Forces : $Q_1 = T_{in}$ = input torque
$Q_2 = 0$

$$T = \frac{1}{2} I_1 \left({}^{R}\omega^{R_1}\right)^2 + \frac{1}{2} m_2 \left[(r_1+r_2)\left({}^{R}\omega^{R_3}\right)\right]^2 + \frac{1}{2} I_2 \left({}^{R}\omega^{R_2}\right)^2 + \frac{1}{2} I_{3_0} \left({}^{R}\omega^{R_3}\right)^2 + \frac{1}{2} m_p \left(\ell_p \, {}^{R}\omega^{R_3}\right)^2$$

$${}^{R}\omega^{R_1} = \dot{\theta}_1 \qquad {}^{R}\omega^{R_3} = \dot{\theta}_3$$

$${}^{R}\omega^{R_2} = -\,{}^{R}\omega^{R_1}\left(\frac{r_1}{r_2}\right) + {}^{R}\omega^{R_3}\left(\frac{r_1+r_2}{r_2}\right) = -\dot{\theta}_1\left(\frac{r_1}{r_2}\right) + \dot{\theta}_3\left(\frac{r_1+r_2}{r_2}\right)$$

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 (r_1+r_2)^2 \dot{\theta}_3^2 + \frac{1}{2} I_2 \left[-\dot{\theta}_1\left(\frac{r_1}{r_2}\right) + \dot{\theta}_3\left(\frac{r_1+r_2}{r_2}\right)\right]^2 + \frac{1}{2} I_{3_0} \dot{\theta}_3^2 + \frac{1}{2} m_p \ell_p^2 \dot{\theta}_3^2$$

$$V = \left[-m_2 g (r_1+r_2) - m_p g \ell_p - m_3 g \frac{\ell}{2}\right](1-\cos\theta_3)$$

$$Q_1 = T_{in} \qquad Q_2 = 0 \qquad q_1 = \theta_1 \qquad q_2 = \theta_3$$

$\underline{q_1 = \theta_1}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = Q_1$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = \bar{I}_1 \dot{\theta}_1 + \bar{I}_2 \left[-\dot{\theta}_1\left(\frac{r_1}{r_2}\right) + \dot{\theta}_3\left(\frac{r_1 + r_2}{r_2}\right)\right]\left(-\frac{r_1}{r_2}\right)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_1} = \bar{I}_1 \ddot{\theta}_1 + \bar{I}_2\left(-\frac{r_1}{r_2}\right)\left[-\ddot{\theta}_1\left(\frac{r_1}{r_2}\right) + \ddot{\theta}_3\left(\frac{r_1 + r_2}{r_2}\right)\right]$$

$$\frac{\partial V}{\partial \theta_1} = 0 \qquad Q_1 = T_{in}$$

$$\frac{\partial T}{\partial \theta_1} = 0$$

$$\left[\bar{\bar{I}}_1 + \bar{I}_2\left(\frac{r_1}{r_2}\right)^2\right]\ddot{\theta}_1 - \bar{I}_2\left(\frac{r_1}{r_2}\right)\left(\frac{r_1 + r_2}{r_2}\right)\ddot{\theta}_3 = T_{in} \qquad [1]$$

$$\underline{q_2 = \theta_3}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_3}\right) - \frac{\partial T}{\partial \theta_3} + \frac{\partial V}{\partial \theta_3} = Q_2$$

$$\frac{\partial T}{\partial \dot{\theta}_3} = m_2 (r_1 + r_2)^2 \dot{\theta}_3 + \overline{I}_2 \left[-\dot{\theta}_1 \left(\frac{r_1}{r_2}\right) + \dot{\theta}_3 \left(\frac{r_1 + r_2}{r_2}\right)\right]\left(\frac{r_1 + r_2}{r_2}\right)$$
$$+ \underline{I}_{3_0} \dot{\theta}_3 + m_p \ell_p^2 \dot{\theta}_3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_3}\right) = \left[m_2 (r_1 + r_2)^2 + \underline{I}_{3_0} + m_p \ell_p^2 \right] \ddot{\theta}_3$$
$$- \overline{I}_2 \ddot{\theta}_1 \left(\frac{r_1}{r_2}\right)\left(\frac{r_1 + r_2}{r_2}\right) + \overline{I}_2 \left(\frac{r_1 + r_2}{r_2}\right)^2 \ddot{\theta}_3$$

$$\frac{\partial V}{\partial \theta_3} = \left[-m_2 g (r_1 + r_2) - m_p g \ell_p - m_3 g \frac{\ell}{2} \right] \sin \theta_3 \qquad Q_2 = 0$$

$$\frac{\partial T}{\partial \theta_3} = 0$$

$$\left[m_2(r_1+r_2)^2 + I_{3_0} + m_p \ell_p^2 + \bar{I}_2\left(\frac{r_1+r_2}{r_2}\right)^2\right]\ddot{\theta}_3$$

$$- \bar{\bar{I}}_2\left(\frac{r_1}{r_2}\right)\left(\frac{r_1+r_2}{r_2}\right)\ddot{\theta}_1 + \left[-m_2 g(r_1+r_2) - m_p g \ell_p - m_3 g \frac{\ell}{2}\right]\sin\theta_3$$

$$= 0 \qquad [2]$$

Summary

$$\left[\bar{I}_1 + \bar{I}_2\left(\frac{r_1}{r_2}\right)^2\right]\ddot{\theta}_1 - \left[\bar{\bar{I}}_2\left(\frac{r_1}{r_2}\right)\left(\frac{r_1+r_2}{r_2}\right)\right]\ddot{\theta}_3 = T_{in} \qquad [1]$$

$$-\left[\bar{\bar{I}}_2\left(\frac{r_1}{r_2}\right)\left(\frac{r_1+r_2}{r_2}\right)\right]\ddot{\theta}_1 + \left[m_2(r_1+r_2)^2 + I_{3_0} + m_p \ell_p^2 + \bar{I}_2\left(\frac{r_1+r_2}{r_2}\right)^2\right]\ddot{\theta}_3$$

$$-\left[m_2 g(r_1+r_2) + m_p g \ell_p + m_3 g \frac{\ell}{2}\right]\sin\theta_3 = 0$$

$$[2]$$

Define Constants:

$$C_1 = \bar{I}_1 + \bar{I}_2 \left(\frac{r_1}{r_2}\right)^2$$

$$C_2 = \bar{I}_2 \left(\frac{r_1}{r_2}\right)\left(\frac{r_1+r_2}{r_2}\right)$$

$$C_3 = m_2 (r_1+r_2)^2 + I_{3_O} + m_p \ell_p^2 + \bar{I}_2 \left(\frac{r_1+r_2}{r_2}\right)^2$$

$$C_4 = m_2 g (r_1+r_2) + m_p g \ell_p + m_3 g \frac{\ell}{2}$$

$$C_1 \ddot{\theta}_1 - C_2 \ddot{\theta}_3 = T_{in} \qquad [1]$$

$$-C_2 \ddot{\theta}_1 + C_3 \ddot{\theta}_3 - C_4 \sin\theta_3 = 0 \qquad [2]$$

Parameters:

I_{3_O} = pendulum rod mass moment of inertia w/r point O

m_2 = planetary gear mass

r_1, r_2 = gear radii

$\bar{I}_1, \bar{I}_2$ = mass moments of inertia of gears about center

m_3, ℓ = pendulum rod mass + length

m_p, ℓ_p = pendulum mass + location w/r to O

Sample Parameter Values:

Actual Hardware Values

$m_p = .01$ kg (0.1 kg)

$\ell = \ell_p = .16$ m (0.330 m)

$r_1 = r_2 = .013$ m (0.0513 m)

$m_1 = m_2 = .061$ kg (0.327 kg)

$m_3 = .04$ kg (0.181 kg)

$\bar{I}_1 = \bar{I}_2 = 5.558\text{E}{-6}$ kg·m^2 $(4.407\text{E}{-4} \text{ kg-m}^2)$

$I_{3_0} = \frac{1}{3} m_3 \ell^2$ $(1.969\text{E}{-4} \text{ kg-m}^2)$

Linearized Equations

$$C_1 \ddot{\theta}_1 - C_2 \ddot{\theta}_3 = T_{in} \qquad [1]$$

$$-C_2 \ddot{\theta}_1 + C_3 \ddot{\theta}_3 - C_4 \theta_3 = 0 \qquad [2]$$

$$\ddot{\theta}_1 = \frac{1}{C_1}\left[T_{in} + C_2 \ddot{\theta}_3\right] \longrightarrow \text{substitute into } [2]$$

$$\ddot{\theta}_3 = \frac{1}{C_3}\left[C_2 \ddot{\theta}_1 + C_4 \theta_3\right] \longrightarrow \text{substitute into } [1]$$

$$C_1 \ddot{\theta}_1 - C_2 \frac{1}{C_3}\left[C_2 \ddot{\theta}_1 + C_4 \theta_3\right] = T_{in}$$

$$-C_2 \frac{1}{C_1}\left[T_{in} + C_2 \ddot{\theta}_3\right] + C_3 \ddot{\theta}_3 - C_4 \theta_3 = 0$$

$$\left[C_1 - \frac{C_2^2}{C_3}\right]\ddot{\theta}_1 - \frac{C_2 C_4}{C_3}\theta_3 = T_{in} \qquad [3]$$

$$\left[C_3 - \frac{C_2^2}{C_1}\right]\ddot{\theta}_3 - C_4 \theta_3 = \frac{C_2}{C_1} T_{in} \qquad [4]$$

State Variables

$q_1 = \theta_1 \qquad q_3 = \theta_3$

$q_2 = \dot{\theta}_1 \qquad q_4 = \dot{\theta}_3$

State Variable Equations

$$\dot{q}_1 = q_2$$

$$\dot{q}_2 = \left(\frac{C_3}{C_3 C_1 - C_2^2}\right)\left[T_{in} + \frac{C_2 C_4}{C_3} q_3\right]$$

$$\dot{q}_3 = q_4$$

$$\dot{q}_4 = \left(\frac{C_1}{C_1 C_3 - C_2^2}\right)\left[\frac{C_2}{C_1} T_{in} + C_4 q_3\right]$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{C_2 C_4}{C_3 C_1 - C_2^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{C_1 C_4}{C_1 C_3 - C_2^2} & 0 \end{bmatrix}}_{A} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{C_3}{C_3 C_1 - C_2^2} \\ 0 \\ \frac{C_2}{C_1 C_3 - C_2^2} \end{bmatrix}}_{B} \begin{bmatrix} T_{in} \end{bmatrix}$$

Transfer functions

$$C_1 \ddot{\theta}_1 - C_2 \ddot{\theta}_3 = T_{in} \qquad [1]$$

$$-C_2 \ddot{\theta}_1 + C_3 \ddot{\theta}_3 - C_4 \theta_3 = 0 \qquad [2]$$

Take Laplace Transform

$$C_1 s^2 \Theta_1 - C_2 s^2 \Theta_3 = T_{in}$$

$$-C_2 s^2 \Theta_1 + C_3 s^2 \Theta_3 - C_4 \Theta_3 = 0$$

$$-C_2 s^2 \Theta_1 + (C_3 s^2 - C_4)\Theta_3 = 0$$

$$\frac{\Theta_1}{\Theta_3} = \frac{C_3 s^2 - C_4}{C_2 s^2}$$

$$C_1 s^2 \left[\frac{C_3 s^2 - C_4}{C_2 s^2} \Theta_3 \right] - C_2 s^2 \Theta_3 = T_{in}$$

$$\frac{C_1 C_3 s^4 - C_1 C_4 s^2}{C_2 s^2} \Theta_3 - C_2 s^2 \Theta_3 = T_{in}$$

$$\frac{(C_1 C_3 - C_2^2) s^4 - C_1 C_4 s^2}{C_2 s^2} \Theta_3 = T_{in}$$

$$\frac{\Theta_3}{T_{in}} = \frac{C_2 s^2}{(C_1 C_3 - C_2^2) s^4 - C_1 C_4 s^2} = \frac{C_2}{(C_1 C_3 - C_2^2) s^2 - C_1 C_4}$$

$$\left.\begin{aligned} \frac{\Theta_3}{T_{in}} &= \frac{C_2}{(C_1 C_3 - C_2^2) s^2 - C_1 C_4} \\[2ex] \frac{\Theta_1}{\Theta_3} &= \frac{C_3 s^2 - C_4}{C_2 s^2} \end{aligned}\right\} \text{Transfer functions}$$

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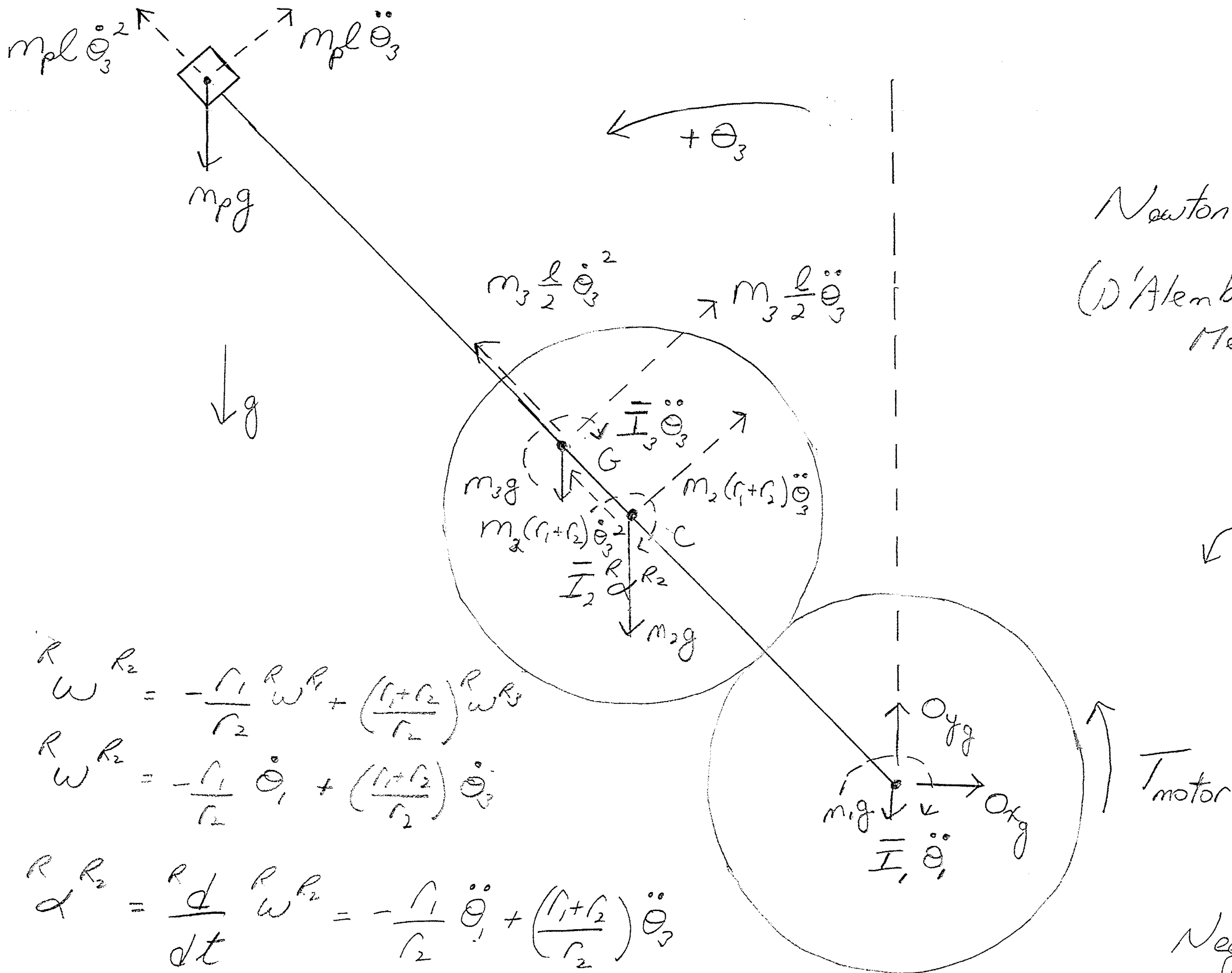
Newton-Euler Method
(D'Alembert's Method)



$${}^R\omega^{R_2} = -\frac{r_1}{r_2}\,{}^R\omega^{R_1} + \left(\frac{r_1+r_2}{r_2}\right){}^R\omega^{R_3}$$

$${}^R\omega^{R_2} = -\frac{r_1}{r_2}\dot{\theta}_1 + \left(\frac{r_1+r_2}{r_2}\right)\dot{\theta}_3$$

$${}^R\alpha^{R_2} = \frac{{}^R d}{dt}\,{}^R\omega^{R_2} = -\frac{r_1}{r_2}\ddot{\theta}_1 + \left(\frac{r_1+r_2}{r_2}\right)\ddot{\theta}_3$$

Neglect friction

$$\sum M_O = 0 \quad \circlearrowleft +$$

$$-\bar{\bar{I}}_1 \ddot{\theta}_1 - \bar{\bar{I}}_2 \alpha^{R_2} - \bar{\bar{I}}_3 \ddot{\theta}_3 - m_2 (r_1+r_2)^2 \ddot{\theta}_3 - m_3 \left(\frac{\ell}{2}\right)^2 \ddot{\theta}_3 - m_p \ell^2 \ddot{\theta}_3 + T_m$$

$$+ m_p g \ell \sin\theta_3 + m_3 g \frac{\ell}{2} \sin\theta_3 + m_2 g (r_1+r_2) \sin\theta_3 = 0$$

$$-\bar{\bar{I}}_1 \ddot{\theta}_1 - \bar{\bar{I}}_2 \left[-\frac{r_1}{r_2} \ddot{\theta}_1 + \frac{r_1+r_2}{r_2} \ddot{\theta}_3 \right] - \bar{\bar{I}}_3 \ddot{\theta}_3 - m_2 (r_1+r_2)^2 \ddot{\theta}_3 - m_3 \left(\frac{\ell}{2}\right)^2 \ddot{\theta}_3 - m_p \ell^2 \ddot{\theta}_3$$

$$+ T_m + g \sin\theta_3 \left[m_p \ell + m_3 \frac{\ell}{2} + m_2 (r_1+r_2) \right] = 0$$

$$\ddot{\theta}_1 \left[\bar{\bar{I}}_1 - \frac{r_1}{r_2} \bar{\bar{I}}_2 \right] + \ddot{\theta}_3 \left[\bar{\bar{I}}_2 \frac{r_1+r_2}{r_2} + \bar{\bar{I}}_3 + m_2 (r_1+r_2)^2 + m_3 \left(\frac{\ell}{2}\right)^2 + m_p \ell^2 \right]$$

$$- g \sin\theta_3 \left[m_p \ell + m_3 \frac{\ell}{2} + m_2 (r_1+r_2) \right] = T_m$$

$$\sum f_x = 0 \quad \xrightarrow{+}$$

$$O_x + m_2(r_1+r_2)\ddot{\theta}_3 \cos\theta_3 + m_3 \frac{l}{2}\ddot{\theta}_3 \cos\theta_3 + m_p l \ddot{\theta}_3 \cos\theta_3$$
$$- m_p l \dot{\theta}_3^2 \sin\theta_3 - m_3 \frac{l}{2}\dot{\theta}_3^2 \sin\theta_3 - m_2(r_1+r_2)\dot{\theta}_3^2 \sin\theta_3 = 0$$

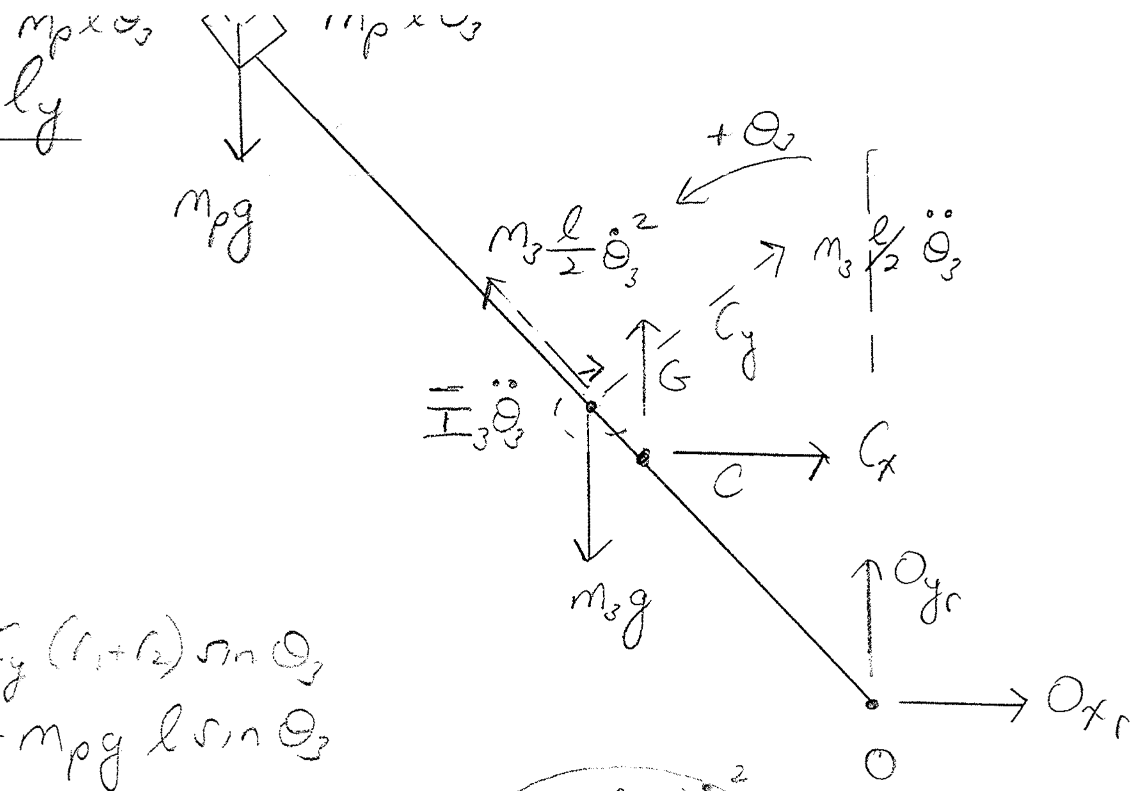
$$O_x + \ddot{\theta}_3 \cos\theta_3 \left[m_2(r_1+r_2) + m_3\frac{l}{2} + m_p l\right]$$
$$- \dot{\theta}_3^2 \sin\theta_3 \left[m_p l + m_3 \frac{l}{2} + m_2(r_1+r_2)\right] = 0$$

$$\sum f_y = 0 \quad \uparrow +$$

$$O_y - m_1 g - m_2 g - m_3 g - m_p g + m_2(r_1+r_2)\ddot{\theta}_3 \sin\theta_3 + m_3\frac{l}{2}\ddot{\theta}_3 \sin\theta_3$$
$$+ m_p l \ddot{\theta}_3 \sin\theta_3 + m_2(r_1+r_2)\dot{\theta}_3^2 \cos\theta_3 + m_3\left(\frac{l}{2}\right)\dot{\theta}_3^2 \cos\theta_3 + m_p l \dot{\theta}_3^2 \cos\theta_3 = 0$$

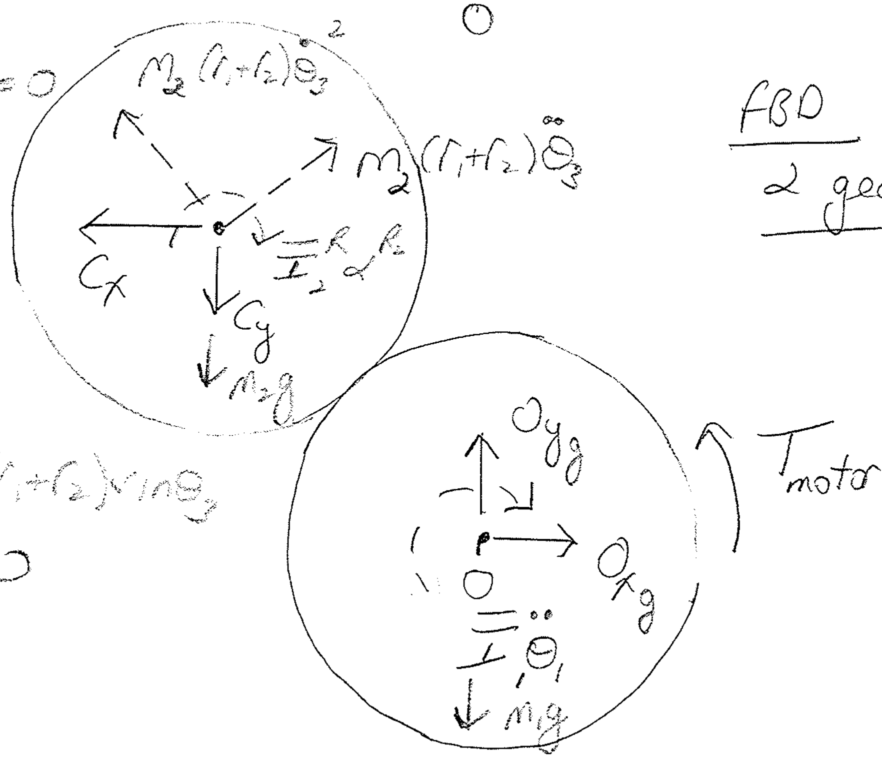
$$O_y - g(m_1+m_2+m_3+m_p) + \ddot{\theta}_3 \sin\theta_3 \left[m_2(r_1+r_2) + m_3\frac{l}{2} + m_p l\right]$$
$$+ \dot{\theta}_3^2 \cos\theta_3 \left[m_p l + m_3\frac{l}{2} + m_2(r_1+r_2)\right] = 0$$

FBD Rod Only



$$\sum M_O = 0$$

$$-C_x(r_1+r_2)\cos\theta_3 - C_y(r_1+r_2)\sin\theta_3 + m_3 g \frac{l}{2}\sin\theta_3 + m_p g\, l \sin\theta_3 - m_3\left(\frac{l}{2}\right)^2 \ddot{\theta}_3 - m_p l^2 \ddot{\theta}_3 - \bar{I}_3 \ddot{\theta}_3 = 0$$



FBD 2 gear

$$\sum M_O = 0$$

$$-\bar{I}_1\ddot{\theta}_1 - \bar{I}_2 \alpha^{R_2} - m_2(r_1+r_2)^2\ddot{\theta}_3 + m_2 g(r_1+r_2)\sin\theta_3 + C_x(r_1+r_2)\cos\theta_3 + C_y(r_1+r_2)\sin\theta_3 + T_m = 0$$

<u>Combine equations:</u>

$$-\bar{I}_1\ddot{\theta}_1 - \bar{I}_2\left[-\frac{r_1}{r_2}\ddot{\theta}_1 + \left(\frac{r_1+r_2}{r_2}\right)\ddot{\theta}_3\right] - m_2(r_1+r_2)^2\ddot{\theta}_3 + m_2 g(r_1+r_2)\sin\theta_3$$
$$+ m_3 g\frac{l}{2}\sin\theta_3 + m_p g\, l\sin\theta_3 - m_3\left(\frac{l}{2}\right)^2\ddot{\theta}_3 - m_p l^2\ddot{\theta}_3 - \bar{I}_3\ddot{\theta}_3$$
$$+ T_m = 0$$

$$\left[\bar{I}_1 - \bar{I}_2\left(\frac{r_1}{r_2}\right)\right]\ddot{\theta}_1 + \ddot{\theta}_3\left[\bar{I}_2\left(\frac{r_1+r_2}{r_2}\right) + m_2(r_1+r_2)^2 + m_3\left(\frac{l}{2}\right)^2 + m_p l^2 + \bar{I}_3\right]$$
$$- g\sin\theta_3\left[m_2(r_1+r_2) + m_3\frac{l}{2} + m_p l\right] = T_m$$

Same as
eq. on pg. 2.