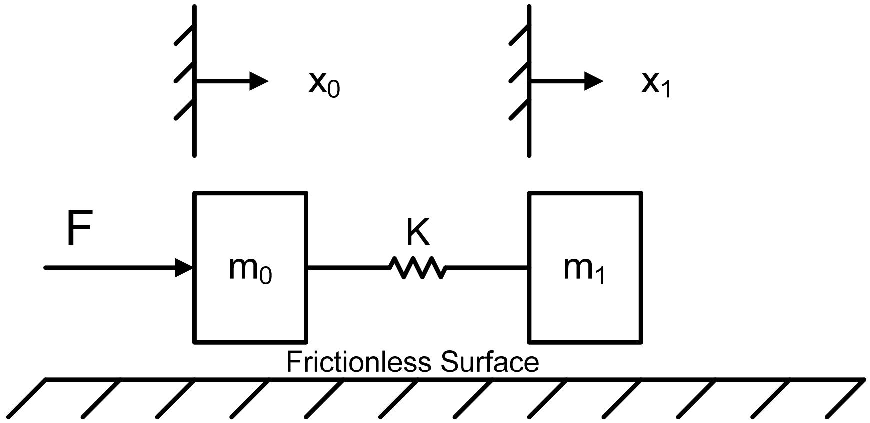
Poles and Zeros

- Mathematically, what are poles and zeros?
- Do poles and zeros have a physical meaning? If so, what is it?
- Do poles and zeros just occur, or does the designer determine the locations of the poles and zeros?
- If this is the case, what exactly does the designer do that determines the location of the poles and zeros?
- Why should a design engineer care where the poles and zeros are located?

Consider the following simple system:



$$m_t = m_0 + m_1$$

$$m_e = \frac{1}{\frac{1}{m_0} + \frac{1}{m_1}}$$

Colocated System

$$G_0(s) = \frac{x_0(s)}{F(s)} = \frac{m_1 s^2 + K}{m_t s^2 (m_e s^2 + K)}$$

Noncolocated System

$$G_1(s) = \frac{x_1(s)}{F(s)} = \frac{K}{m_t s^2 (m_e s^2 + K)}$$

– A pole of a transfer function is a value of *s* that makes the denominator equal to zero.

– A zero of a transfer function is a value of *s* that makes the numerator of equal to zero.

Open-Loop Poles

$$s = 0 \qquad s = 0 \qquad s = \pm i\sqrt{\frac{K}{m_e}}$$

Open-Loop Zeros

Colocated System: $s = \pm i\sqrt{\frac{K}{m_1}}$

Noncolocated System: No Zeros

Natural Frequencies & Mode Shapes

Rigid Body Mode $\omega = 0$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Flexible Mode $\omega = \sqrt{\frac{K}{m_e}}$ $\begin{bmatrix} \frac{m_1}{m_0} \\ -1 \end{bmatrix}$

Colocated Transfer Function

$$G(s) = \frac{x_2(s)}{F(s)} = \frac{Ms^2 + 2K}{(Ms^2 + 3K)(Ms^2 + K)}$$

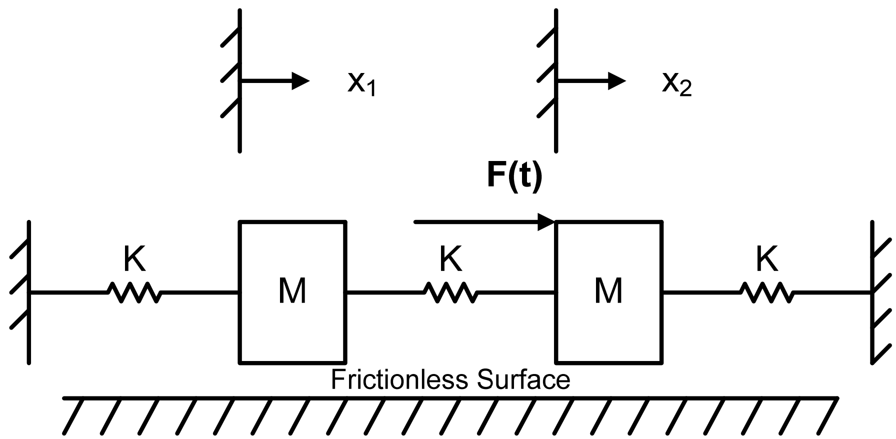
Complex Conjugate Poles

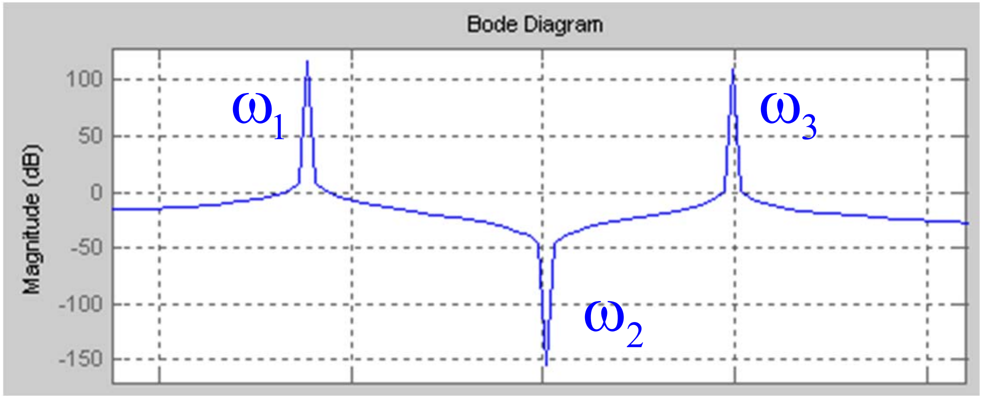
$$\pm i\omega_1 \qquad \omega_1 = \sqrt{\frac{K}{M}}$$

$$\pm i\omega_3 \qquad \omega_3 = \sqrt{\frac{3K}{M}}$$

Complex Conjugate Zeros

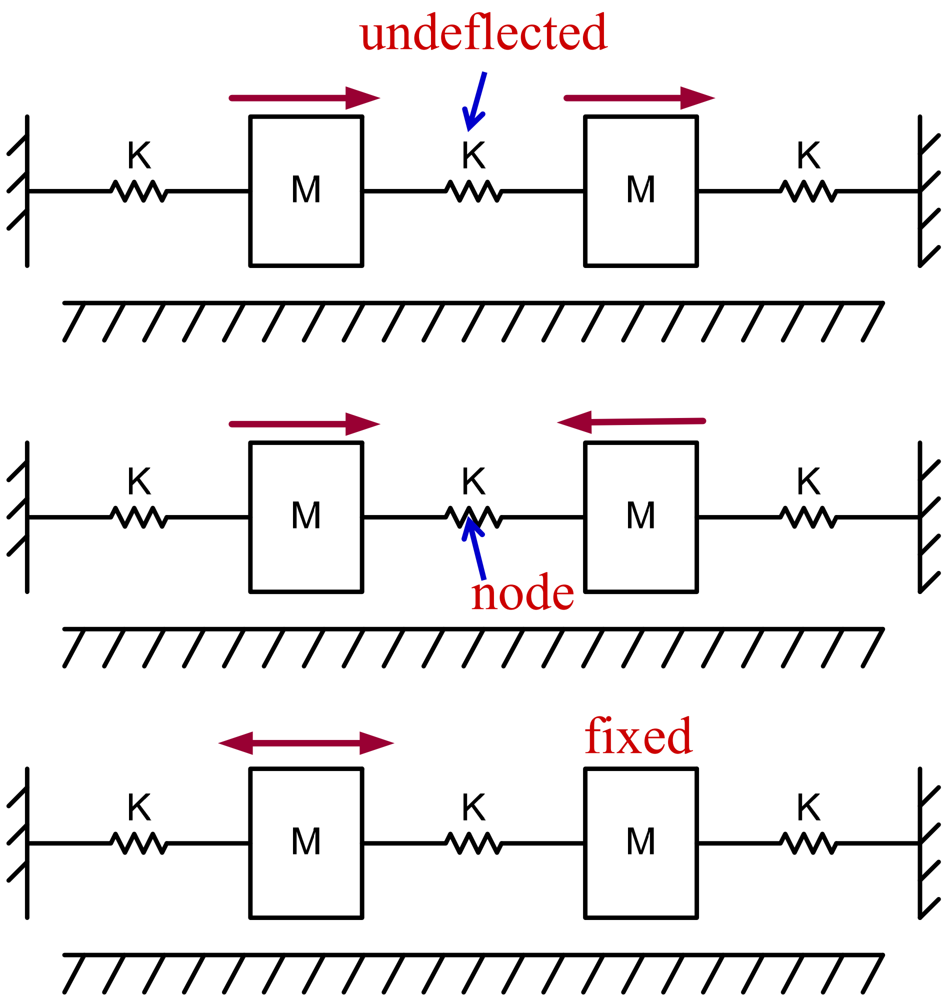
$$\pm i\omega_2 \qquad \omega_1 = \sqrt{\frac{2K}{M}}$$





Consider This System

Mode Shapes

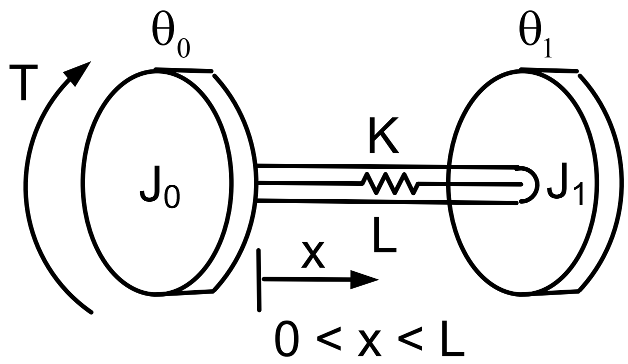


$$\omega_1 = \sqrt{\frac{K}{M}}$$

$$\omega_3 = \sqrt{\frac{3K}{M}}$$

$$\omega_2 = \sqrt{\frac{2K}{M}}$$

- Migration of Zeros
  - Consider two solid disks connected with an elastic rod.



$$G_0(s) = \frac{\theta_0(s)}{T(s)} = \frac{J_1 s^2 + K}{s^2[J_0 J_1 s^2 + K(J_1 + J_0)]} \qquad G_1(s) = \frac{\theta_1(s)}{T(s)} = \frac{K}{s^2[J_0 J_1 s^2 + K(J_1 + J_0)]}$$

  - Suppose a sensor is connected at x = cL where c varies from 0 to 1. Then the measurement θ_c would be a linear combination of θ_0 and θ_1: $\theta_c = (1-c)\theta_0 + c\theta_1$

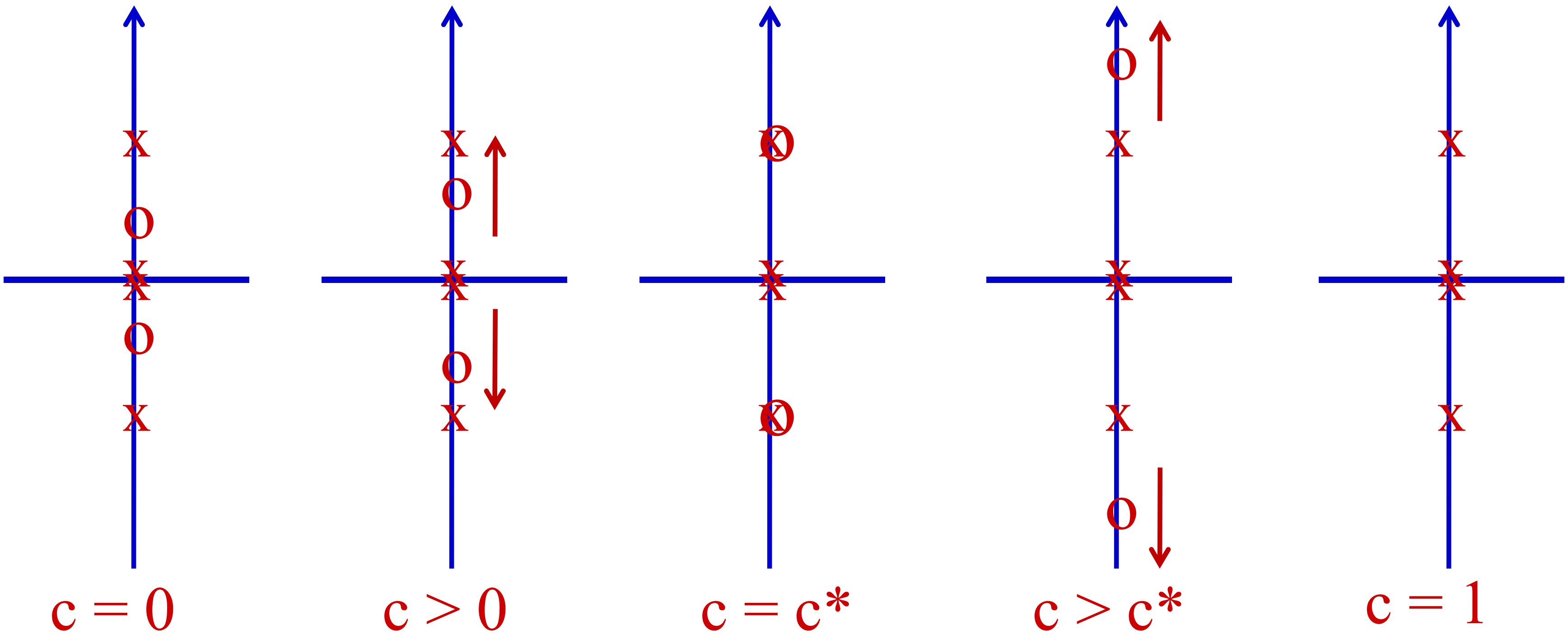
$$\frac{\theta_c(s)}{T(s)} = \frac{(1-c)\theta_0(s)}{T(s)} + \frac{c\theta_1(s)}{T(s)} = \frac{(1-c)J_1 s^2 + K}{s^2[J_0 J_1 s^2 + K(J_1 + J_0)]}$$

poles

$s = 0 \qquad s = 0 \qquad s = \pm i\sqrt{\dfrac{K(J_0 + J_1)}{J_0 J_1}}$

zeros

$s = \pm i\sqrt{\dfrac{K}{(1-c)J_1}}$



Location of zeros as a function of c

$$c^* = \frac{J_1}{J_0 + J_1}$$

– What happens physically when the location of the sensor is such that $c = c^*$ or when there is pole-zero cancellation?

- This corresponds to the situation where the sensor is located at the node of the flexible mode (there is only one here) and you are therefore not able to detect the flexible mode, as seen here:

$$\frac{\theta_c(s)}{T(s)} = \frac{(1-c^*)J_1 s^2 + K}{s^2[J_0 J_1 s^2 + K(J_1 + J_0)]} = \frac{\left(\dfrac{J_0 J_1}{J_0 + J_1}\right)s^2 + K}{s^2[J_0 J_1 s^2 + K(J_1 + J_0)]} = \frac{1}{(J_0 + J_1)s^2}$$

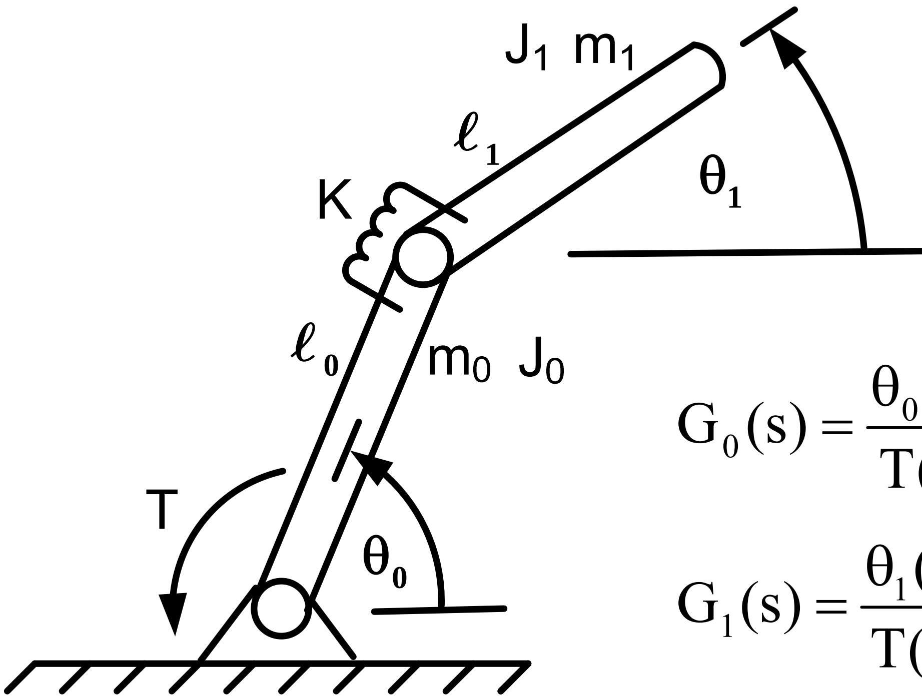
- So we see that when $c = c^*$, the resonant frequency of the constrained subsystem is exactly the same as that of the original system such that the zeros are the same as the poles.

- Let's summarize what we have found:
  - Complex-conjugate poles are the resonant frequencies of the flexible system.
  - Complex-conjugate zeros are the resonant frequencies of the subsystem constrained by the sensor and actuator.
  - For both discrete and continuous systems, when the sensor and actuator are colocated, poles and zeros will alternate along the imaginary axis. As the sensor is moved away from the actuator, the poles do not change since they are independent of the location of the sensors and actuators.
  - But the zeros will migrate along the imaginary axis toward infinity. Along the way, whenever the sensor is located at the nodal point of a particular flexible mode, there will be pole-zero cancellation.

- Nonminimum Phase Systems
  - Systems that have no poles or zeros in the RHP are called minimum-phase systems, as either of two components of the frequency response, gain and phase, contains all the frequency response information that exists. This is called Bode's Gain-Phase relationship.
  - Here we focus on nonminimum-phase stable systems (no poles in the RHP), i.e., systems with zeros in the RHP.
  - Physical phenomena that give rise to nonminimum-phase behavior include:
    - Control of the level of a volume of boiling water.
    - Hydroelectricity generation.
    - Sequences of interacting processes.

- <u>Nonminimum Phase System Example</u>
  - Consider the following system of two rigid links and a torsional spring. Assume small displacements.

$$\begin{bmatrix} J_0 + m_1\ell_0^2 & \ell_0\ell_1 m_1 \\ \ell_0\ell_1 m_1 & J_1 + m_1\ell_1^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}\begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} T \\ 0 \end{bmatrix}$$



$$J_t = J_0 + J_1 + m_1(\ell_0 + \ell_1)^2$$

$$J_e = \frac{1}{J_t}[J_0 J_1 + m_1(J_0\ell_1^2 + J_1\ell_0^2)]$$

$$G_0(s) = \frac{\theta_0(s)}{T(s)} = \frac{(J_1 + m_1\ell_1^2)s^2 + K}{J_t s^2(J_e s^2 + K)}$$

$$G_1(s) = \frac{\theta_1(s)}{T(s)} = \frac{-\ell_0\ell_1 m_1 s^2 + K}{J_t s^2(J_e s^2 + K)}$$

<u>Poles</u>
rigid body mode
+ flexible mode

– Zeros of the System

- Colocated System
  – There is a complex-conjugate zero pair on the imaginary axis between the poles.

$$s = \pm i\sqrt{\frac{K}{J_1 + m_1\ell_1^2}} < \pm i\sqrt{\frac{K}{J_e}}$$

  – The zeros correspond to the resonant frequencies of the constrained system, i.e., the resonant frequencies of the second link when the first link is fixed.
- Noncolocated System
  – There are a pair of real zeros.

$$s = \pm\sqrt{\frac{K}{\ell_0\ell_1 m_1}}$$

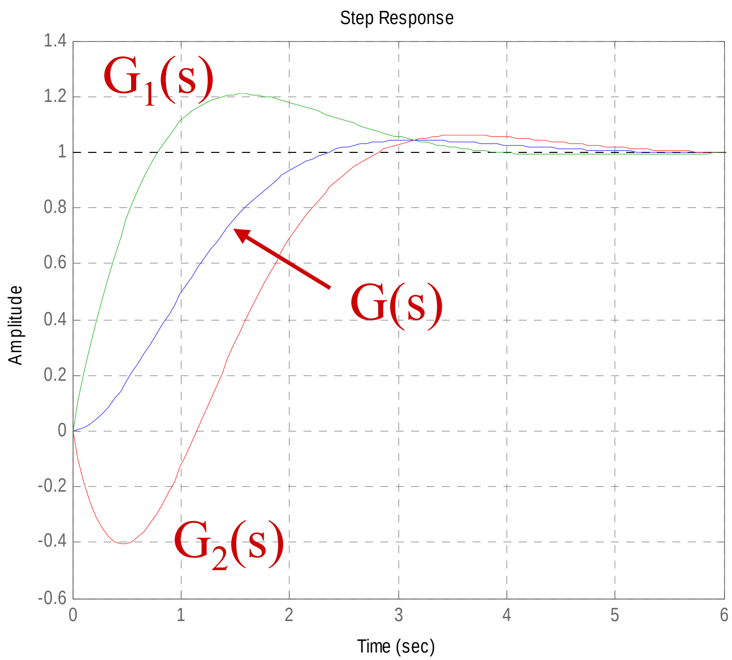
  – A zero in the right half plane is called a nonminimum phase zero and has important significance in control.
  – This is a nonminimum phase discrete system.

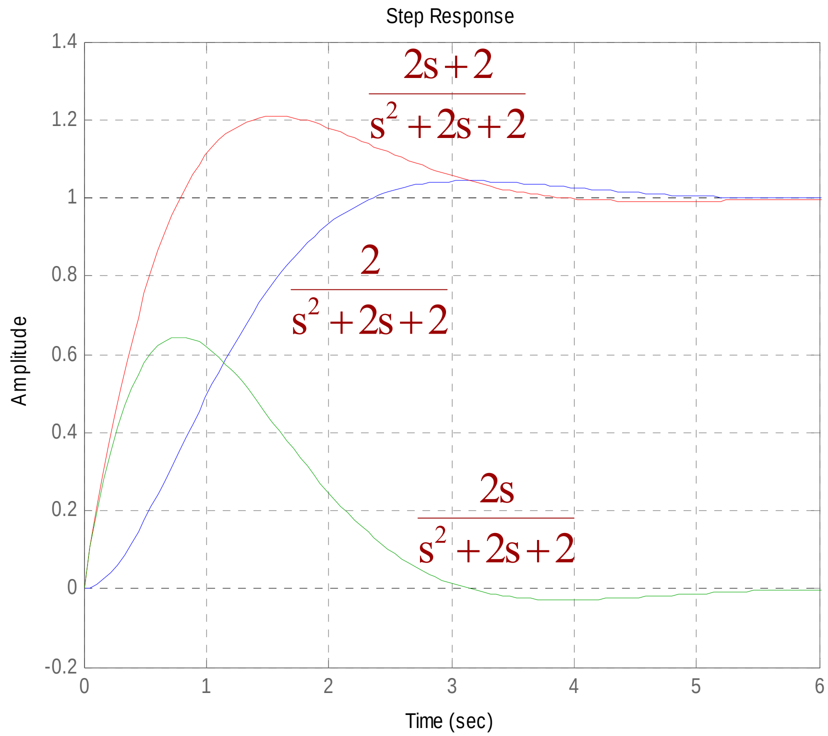
Effect of a Positive Real Zero

$$G(s) = \frac{2}{\left(s^2 + 2s + 2\right)}$$

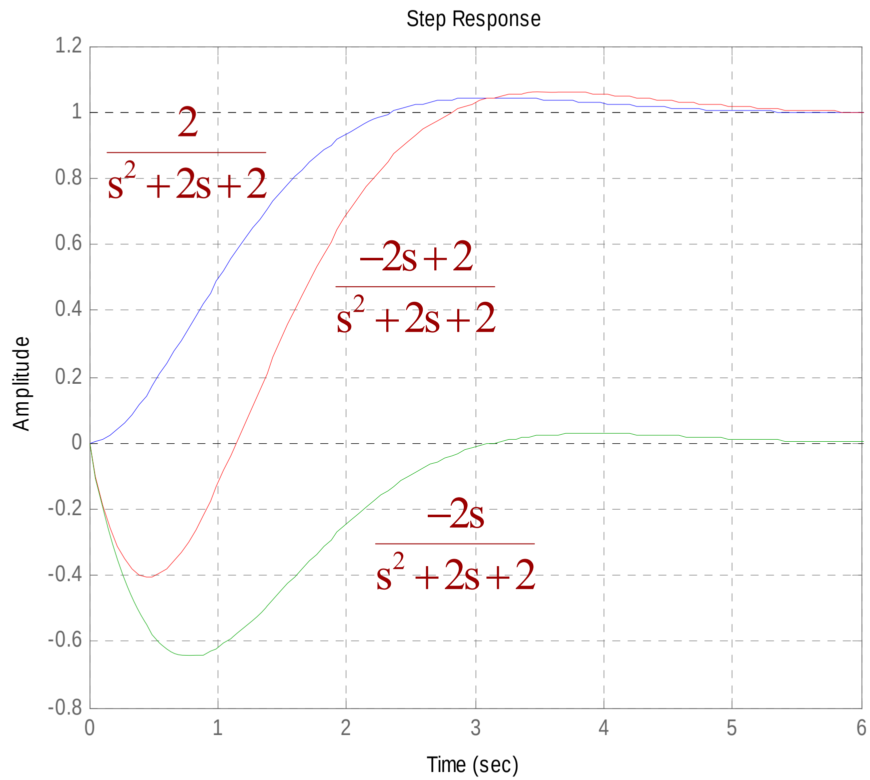
$$G_1(s) = \frac{2}{\left(s^2 + 2s + 2\right)} + \frac{2s}{\left(s^2 + 2s + 2\right)} = \frac{\left(2s + 2\right)}{\left(s^2 + 2s + 2\right)}$$ G(s) plus its derivative

$$G_2(s) = \frac{2}{\left(s^2 + 2s + 2\right)} - \frac{2s}{\left(s^2 + 2s + 2\right)} = \frac{\left(-2s + 2\right)}{\left(s^2 + 2s + 2\right)}$$ G(s) minus its derivative





Step Response
$\frac{2s+2}{s^2+2s+2}$
$\frac{2}{s^2+2s+2}$
$\frac{2s}{s^2+2s+2}$
Amplitude
Time (sec)



Step Response
$\frac{2}{s^2+2s+2}$
$\frac{-2s+2}{s^2+2s+2}$
$\frac{-2s}{s^2+2s+2}$
Amplitude
Time (sec)

- $G_1(s)$ has a pair of real zeros.
- For $\theta_1(t) = 0$, T(t) must be such that the net torque applied to the second link equals zero. For a given T(t), there will be a nonzero $\theta_0(t) = 0$, and in order for $\theta_1(t)$ to be zero in spite of $\theta_0(t)$, the second link must rotate in the opposite direction relative to the first link so that at the joint connecting the two links, there will be a restoring torque ($K\theta_0$) due to the rotational spring.
- Since $\theta_1(t) = 0$, and $\theta_1(t)$ is the absolute angular displacement, then link 2 is in rigid-body translation with an absolute acceleration equal to that of the joint: $\ell_0 \ddot{\theta}_0$
- Furthermore, for the second link to have zero rotational acceleration, the spring torque must equal the inertia torque due to $m_1 a_1$ such that:

$$K\theta_0 = m_1 a \ell_1 \cos\theta_1 \approx m_1 \ell_0 \ddot{\theta}_0 \ell_1 \quad (\text{small } \theta_1)$$