Slider-Crank Mechanism
– Kinematics and Kinetics

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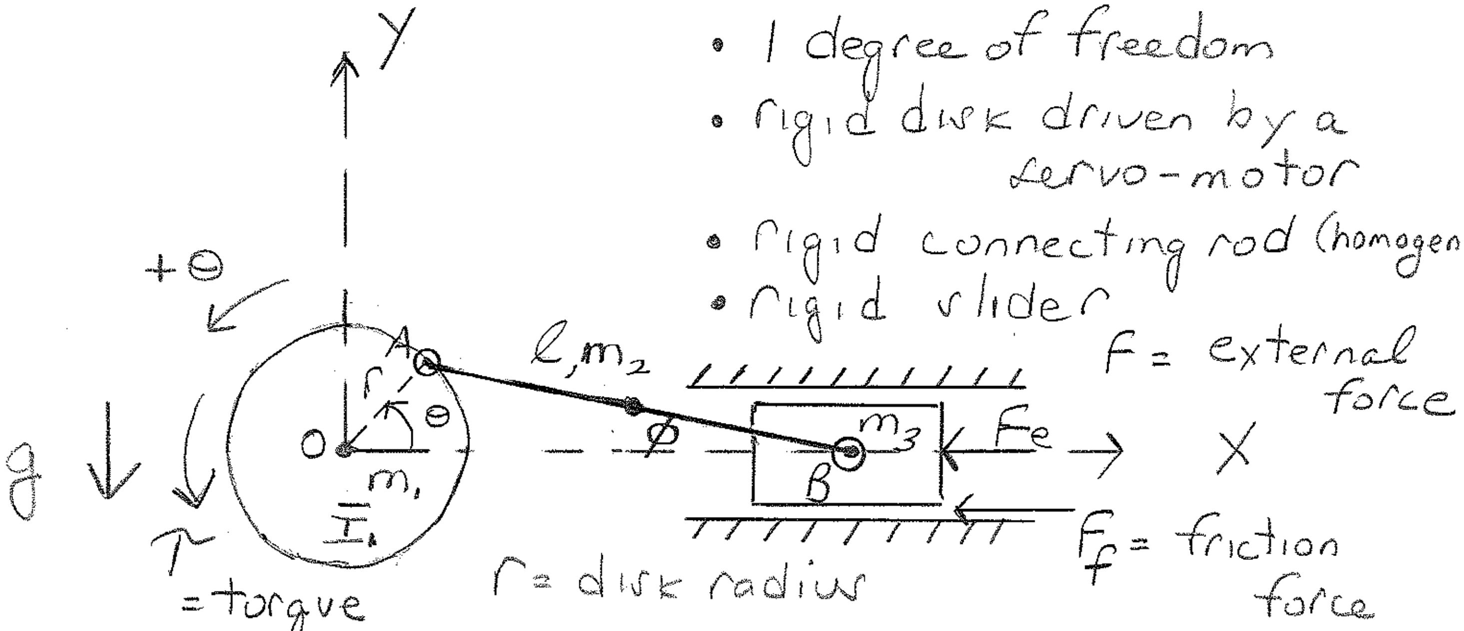
6-19-2012

- 1 degree of freedom
- rigid disk driven by a servo-motor
- rigid connecting rod (homogeneous)
- rigid slider



Locations of CGs:

$$\bar{x}_1 = 0 \quad \bar{y}_1 = 0 \qquad [1]$$

$$\left.\begin{aligned} \bar{x}_2 &= r\cos\theta + \frac{l}{2}\cos\phi \\ \bar{y}_2 &= \frac{l}{2}\sin\phi \end{aligned}\right\} \qquad [2]$$

$$\bar{x}_3 = r\cos\theta + l\cos\phi \qquad \bar{y}_3 = 0 \qquad [3]$$

Constraint Equation

$$r\sin\theta = l\sin\phi \qquad [4]$$

$$\phi = \sin^{-1}\left(\frac{r}{l}\sin\theta\right)$$

$$[5]$$

Kinematic Analysis

$$x_B = r\cos\theta + l\cos\phi$$

$$\dot{x}_B = -r\dot{\theta}\sin\theta - l\dot{\phi}\sin\phi \qquad [6]$$

$$\ddot{x}_B = -r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - l\ddot{\phi}\sin\phi - l\dot{\phi}^2\cos\phi \qquad [7]$$

$$\sin\phi = \frac{r}{l}\sin\theta$$

$$\dot{\phi}\cos\phi = \frac{r}{l}\dot{\theta}\cos\theta$$

$$\dot{\phi} = \frac{r\dot{\theta}\cos\theta}{l\cos\phi} \qquad [8]$$

$$\ddot{\phi} = \frac{r\ddot{\theta}\cos\theta\cos\phi + r\dot{\theta}\dot{\phi}\cos\theta\sin\phi - r\dot{\theta}^2\sin\theta\cos\phi}{l\cos^2\phi} \qquad [9]$$

Kinetic Analysis

Lagrange Formulation $\qquad$ 1 DOF

Generalized Coordinate: θ

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_\theta$$

Kinetic Energy $T = T_1 + T_2 + T_3$

$$T_1 = \frac{1}{2}\bar{I}_1 \dot{\theta}^2 \qquad \bar{I}_1 = \frac{1}{2} m_1 r^2$$

r = disk radius

$$= \frac{1}{2}\left(\frac{1}{2} m_1 r^2\right)\dot{\theta}^2$$

$$= \frac{1}{4} m_1 r^2 \dot{\theta}^2$$

$$T_2 = \frac{1}{2}\bar{I}_2 \dot{\phi}^2 + \frac{1}{2} m_2 \bar{V}_2^{\,2} \qquad \bar{I}_2 = \frac{1}{12} m_2 \ell^2$$

$$\bar{V}_2^{\,2} = \dot{x}_2^{\,2} + \dot{y}_2^{\,2}$$

$$= \left[-r\dot{\theta}\sin\theta - \frac{\ell}{2}\dot{\phi}\sin\phi\right]^2 + \left[\frac{\ell}{2}\dot{\phi}\cos\phi\right]^2$$

$$T_2 = \frac{1}{24} m_2 \ell^2 \dot{\phi}^2 + \frac{1}{2} m_2 \left[r^2\dot{\theta}^2\sin^2\theta + r\ell\dot{\theta}\dot{\phi}\sin\theta\sin\phi + \frac{\ell^2}{4}\dot{\phi}^2\sin^2\phi + \frac{\ell^2}{4}\dot{\phi}\cos^2\phi\right]$$

$$T_2 = \frac{1}{6} m_2 \ell^2 \dot{\phi}^2 + \frac{1}{2} m_2 r^2 \dot{\theta}^2 \sin^2\theta + \frac{1}{2} m_2 r\ell\dot{\theta}\dot{\phi}\sin\theta\sin\phi$$

$$T_3 = \frac{1}{2} m_3 \dot{x}_3^2$$

$$= \frac{1}{2} m_3 \left[-r\dot{\theta}\sin\theta - \ell\dot{\phi}\sin\phi \right]^2$$

$$= \frac{1}{2} m_3 r^2 \dot{\theta}^2 \sin^2\theta + m_3 r \ell \dot{\theta}\dot{\phi}\sin\theta\sin\phi$$

$$+ \frac{1}{2} m_3 \ell^2 \dot{\phi}^2 \sin^2\phi$$

$$T = T_1 + T_2 + T_3$$

Potential Energy

$$V = V_1 + V_2 + V_3$$

$$V_1 = 0$$

$$V_2 = m_2 g \left(\frac{\ell}{2} \sin\phi \right) = \frac{1}{2} m_2 g \ell \sin\phi$$

$$V_3 = 0$$

Express T and V in terms of θ.

$$T_1 = \frac{1}{4} m_1 r^2 \dot{\theta}^2$$

$$T_2 = \frac{1}{6} m_2 \ell^2 \dot{\phi}^2 + \frac{1}{2} m_2 r^2 \dot{\theta}^2 \sin^2\theta + \frac{1}{2} m_2 r \ell \dot{\theta} \dot{\phi} \sin\theta \sin\phi$$

Define: $c = \ell \cos\phi$

$$\left\{ \begin{aligned} &(r\sin\theta)^2 + (c)^2 = (\ell)^2 \\ &c = \left[\ell^2 - r^2\sin^2\theta\right]^{1/2} \\ &\cos\phi = \frac{c}{\ell} \\ &\sin\phi = \frac{r}{\ell}\sin\theta \\ &\dot{\phi} = \frac{r\dot{\theta}\cos\theta}{\ell\cos\phi} = \frac{r\dot{\theta}\cos\theta}{c} \end{aligned} \right.$$

$$T_2 = \frac{1}{6} m_2 \ell^2 \left[\frac{r\dot{\theta}\cos\theta}{c}\right]^2$$

$$+ \frac{1}{2} m_2 r^2 \dot{\theta}^2 \sin^2\theta$$

$$+ \frac{1}{2} m_2 r \ell \dot{\theta} \sin\theta \left(\frac{r\dot{\theta}\cos\theta}{c}\right)\left(\frac{r}{\ell}\sin\theta\right)$$

$$T_2 = \frac{1}{6} m_2 \frac{\ell^2 r^2}{c^2} \dot{\theta}^2 \cos^2\theta + \frac{1}{2} m_2 r^2 \dot{\theta}^2 \sin^2\theta$$

$$+ \frac{1}{2} m_2 \frac{r^3}{c} \dot{\theta}^2 \sin^2\theta \cos\theta$$

$$T_2 = m_2 \dot{\theta}^2 \left[\frac{1}{6}\left(\frac{\ell r}{c}\cos\theta\right)^2 + \frac{1}{2}(r\sin\theta)^2 \right.$$

$$\left. + \frac{1}{2}\frac{r^3}{c}\cos\theta\sin^2\theta\right]$$

$$T_3 = \frac{1}{2} m_3 r^2 \dot{\theta}^2 \sin^2\theta + m_3 r \ell \dot{\theta} \dot{\phi} \sin\theta \sin\phi$$

$$+ \frac{1}{2} m_3 \ell^2 \dot{\phi}^2 \sin^2\phi$$

$$T_3 = \frac{1}{2} m_3 r^2 \dot{\theta}^2 \sin^2\theta$$

$$+ m_3 r \ell \dot{\theta} \sin\theta \left(\frac{r\dot{\theta}\cos\theta}{c}\right)\left(\frac{r}{\ell}\sin\theta\right)$$

$$+ \frac{1}{2} m_3 \ell^2 \left(\frac{r\dot{\theta}\cos\theta}{c}\right)^2 \left(\frac{r}{\ell}\sin\theta\right)^2$$

$$T_3 = m_3 \dot{\theta}^2 \left[\frac{1}{2}(r\sin\theta)^2 + \frac{r^3 \sin^2\theta \cos\theta}{c} + \frac{1}{2}\frac{r^4 \sin^2\theta \cos^2\theta}{c^2} \right]$$

$$V_1 = 0 \qquad V_3 = 0$$

$$V_2 = \frac{1}{2} m_2 g \, l \sin\phi = \frac{1}{2} m_2 g \, l \left(\frac{r}{l} \sin\theta \right)$$

$$V_2 = \frac{1}{2} m_2 g r \sin\theta$$

Summary

$$T = \frac{1}{4} m_1 r^2 \dot{\theta}^2 +$$

$$m_2 \dot{\theta}^2 \left[\frac{1}{6}\left(\frac{l r \cos\theta}{c}\right)^2 + \frac{1}{2}(r\sin\theta)^2 + \frac{1}{2}\frac{r^3}{c}\cos\theta\sin^2\theta \right]$$

$$+ m_3 \dot{\theta}^2 \left[\frac{1}{2}(r\sin\theta)^2 + \frac{r^3\sin^2\theta\cos\theta}{c} + \frac{1}{2}\frac{r^4\sin^2\theta\cos^2\theta}{c^2} \right]$$

$$V = \frac{1}{2} m_2 g r \sin\theta$$

$$c = \left[l^2 - r^2 \sin^2\theta \right]^{1/2}$$

Generalized Torque Q_θ

$$x_3 = r\cos\theta + l\cos\phi$$
$$x_3 = r\cos\theta + l\left(\frac{c}{l}\right) = r\cos\theta + c$$

Work done by external torques/forces:

$$\delta W = \tau\,\delta\theta + (f_e + f_f)\,\delta x$$

$$x = r\cos\theta + l\cos\phi = r\cos\theta + c$$
$$\left(c^2 = l^2 - r^2\sin^2\theta\right)$$

$$x = r\cos\theta + \left[l^2 - r^2\sin^2\theta\right]^{1/2}$$

$$\frac{dx}{d\theta} = -r\sin\theta + \frac{1}{2}\left[l^2 - r^2\sin^2\theta\right]^{-1/2}\left(-2r^2\sin\theta\cos\theta\right)$$

$$= -r\sin\theta - \frac{r^2\sin\theta\cos\theta}{c}$$

$$= -r\sin\theta\left[1 + \frac{r\cos\theta}{c}\right]$$

$$dx = -r\sin\theta\left[1 + \frac{r\cos\theta}{c}\right]d\theta$$

$$\delta W = \tau\,\delta\theta + (f_e + f_f)\left[-r\sin\theta\left(1 + \frac{r\cos\theta}{c}\right)\right]\delta\theta$$

$$Q_\theta = \tau - (f_e + f_f)(r\sin\theta)\left(1 + \frac{r\cos\theta}{c}\right)$$

Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = Q_\theta$$

(Details omitted - just too much!)

Result $M(\theta)\,\ddot\theta + N(\theta,\dot\theta) = F(\theta)$

$$M = \left[(2m_3 + m_2) + \frac{m_3}{c}r\cos\theta\right]\left[\frac{r^3}{c}\cos\theta\sin^2\theta\right] + (m_2 + m_3)r^2\sin^2\theta + \frac{1}{3}m_2\left(\frac{\ell}{c}\right)^2(r\cos\theta)^2 + \frac{1}{2}m_1 r^2$$

$$N = \Bigg\{m_2 r^2\sin\theta\cos\theta\left[1 - \frac{\ell^2}{3c^2} + \frac{r}{c}\cos\theta + \frac{(\ell r)^2}{3c^4}\cos^2\theta + \frac{r^3}{2c^3}\cos\theta\sin^2\theta\right] - m_2\frac{r^3}{2c}\sin^3\theta + m_3 r^2\sin\theta\cos\theta\left[1 - \frac{r^2}{c^2}\sin^2\theta + \frac{r^2}{c^2}\cos^2\theta + \frac{2r}{c}\cos\theta + \frac{r^4\cos^2\theta\sin^2\theta}{c^4} + \frac{r^3}{c^3}\sin^2\theta\cos\theta\right] - m_3\frac{r^3}{c}\sin^3\theta\Bigg\}\dot\theta^2 + \frac{1}{2}m_2 g r\cos\theta$$

$$F(\theta) = \tau - (F_e + F_f)(r \sin\theta)\left(1 + \frac{r}{c}\cos\theta\right)$$

$$c = \left[\ell^2 - r^2 \sin^2\theta\right]^{1/2}$$