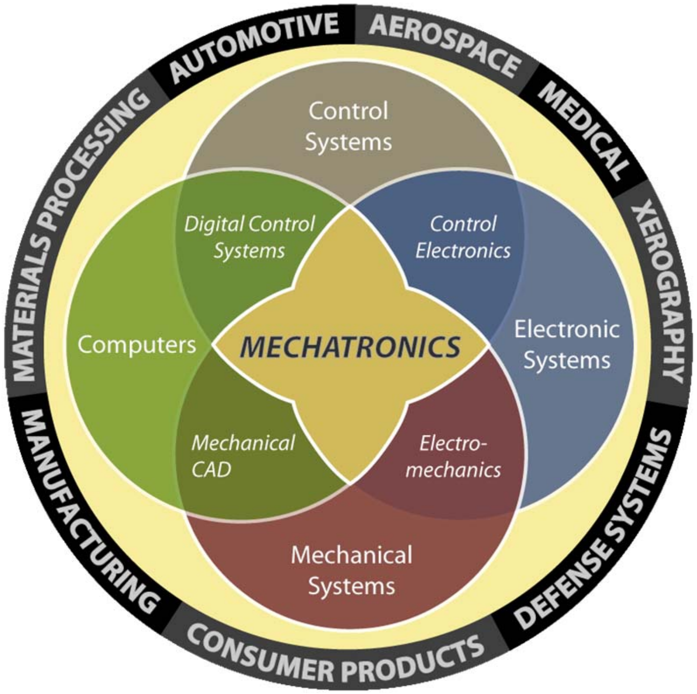
Stability of Feedback Control Systems: Absolute and Relative



Dr. Kevin Craig

Greenheck Chair in Engineering Design &
Professor of Mechanical Engineering

Marquette University

Topics

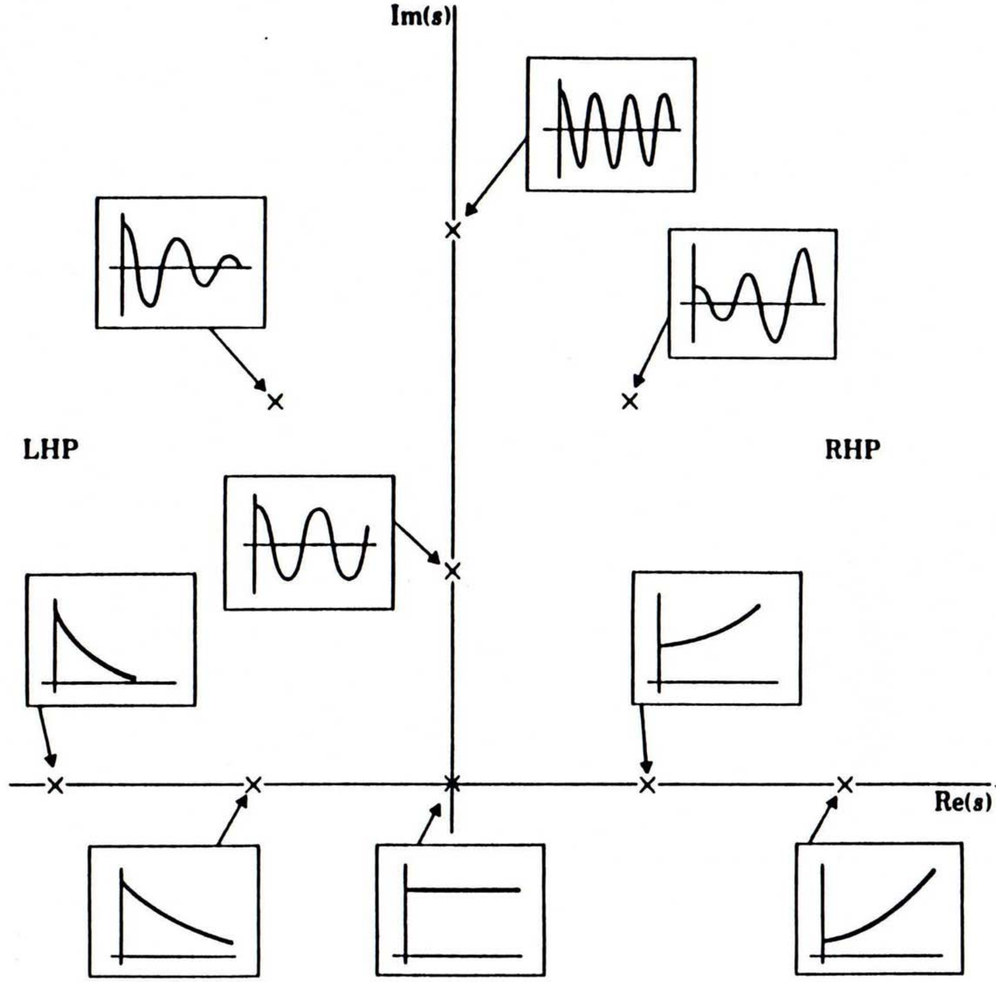
- Introduction
- Routh Stability Criterion
  - Approximating Systems with Time Delay
- Nyquist Stability Criterion
- Root-Locus Interpretation of Stability

Introduction

- If a system in equilibrium is momentarily excited by command and/or disturbance inputs and those inputs are then removed, the system must return to equilibrium if it is to be called absolutely stable.
- If action persists indefinitely after excitation is removed, the system is judged absolutely unstable.
- If a system is stable, how close is it to becoming unstable? Relative stability indicators are gain margin and phase margin.

- If we want to make valid stability predictions, we must include enough dynamics in the system model so that the closed-loop system differential equation is at least third order.
  - An exception to this rule involves systems with dead times, where instability can occur when the dynamics (other than the dead time itself) are zero, first, or second order.

- The analytical study of stability becomes a study of the stability of the solutions of the closed-loop system's differential equations.
- A complete and general stability theory is based on the locations in the complex plane of the *roots of the closed-loop system characteristic equation*, stable systems having all of their roots in the LHP.



Time Functions Associated with Points in the Complex Plane

- Results of practical use to engineers are mainly limited to linear systems with constant coefficients, where an exact and complete stability theory has been known for a long time.
- Exact, general results for linear time-variant and nonlinear systems are nonexistant. Fortunately, the linear time-invariant theory is adequate for many practical systems.
- For nonlinear systems, an approximation technique called the describing function technique has a good record of success.
- Digital simulation is always an option and, while no general results are possible, one can explore enough typical inputs and system parameter values to gain a high degree of confidence in stability for any specific system.

- Two general methods of determining the presence of unstable roots without actually finding their numerical values are:
  - Routh Stability Criterion
    - This method works with the closed-loop system characteristic equation in an algebraic fashion.
  - Nyquist Stability Criterion
    - This method is a graphical technique based on the open-loop frequency response polar plot.
- Both methods give the same results, a statement of the number (but not the specific numerical values) of unstable roots. This information is generally adequate for design purposes.

- This theory predicts excursions of infinite magnitude for unstable systems. Since infinite motions, voltages, temperatures, etc., require infinite power supplies, no real-world system can conform to such a mathematical prediction, casting possible doubt on the validity of our linear stability criterion since it predicts an impossible occurrence.
- What actually happens is that oscillations, if they are to occur, start small, under conditions favorable to and accurately predicted by the linear stability theory. They then start to grow, again following the exponential trend predicted by the linear model. Gradually, however, the amplitudes leave the region of accurate linearization, and the linearized model, together with all its mathematical predictions, loses validity.

- Since solutions of the now nonlinear equations are usually not possible analytically, we must now rely on experience with real systems and/or nonlinear computer simulations when explaining what really happens as unstable oscillations build up.
- First, practical systems often include over-range alarms and safety shut-offs that automatically shut down operation when certain limits are exceeded. If certain safety features are not provided, the system may destroy itself, again leading to a shut-down condition. If safe or destructive shut-down does not occur, the system usually goes into a limit-cycle oscillation, an ongoing, nonsinusoidal oscillation of fixed amplitude. The wave form, frequency, and amplitude of limit cycles is governed by nonlinear math models that are usually analytically unsolvable.

Routh Stability Criterion

- To use the Routh Stability criterion we must have in hand the characteristic equation of the closed-loop system's differential equation.
- Routh's criterion requires the characteristic equation to be a polynomial in the differential operator D. Therefore any dead times must be approximated with polynomial forms in D.

- Systems with Time Delay
  - Time delays or dead-times (DT's) between inputs and outputs are very common in industrial processes, engineering, economical, and biological systems.
  - Transportation and measurement lags, analysis times, computation and communication lags all introduce DT's into control loops.
  - DT's are also used to compensate for model reduction where high-order systems are represented by low-order models with delays.
  - Two major consequences:
    - Complicates the analysis and design of feedback control systems
    - Makes satisfactory control more difficult to achieve

- Any delay in measuring, in controller action, in actuator operation, in computer computation, and the like, is called *transport delay* or *dead time*, and it always reduces the stability of a system and limits the achievable response time of the system.

$$q_i(t) = \text{ input to dead-time element}$$

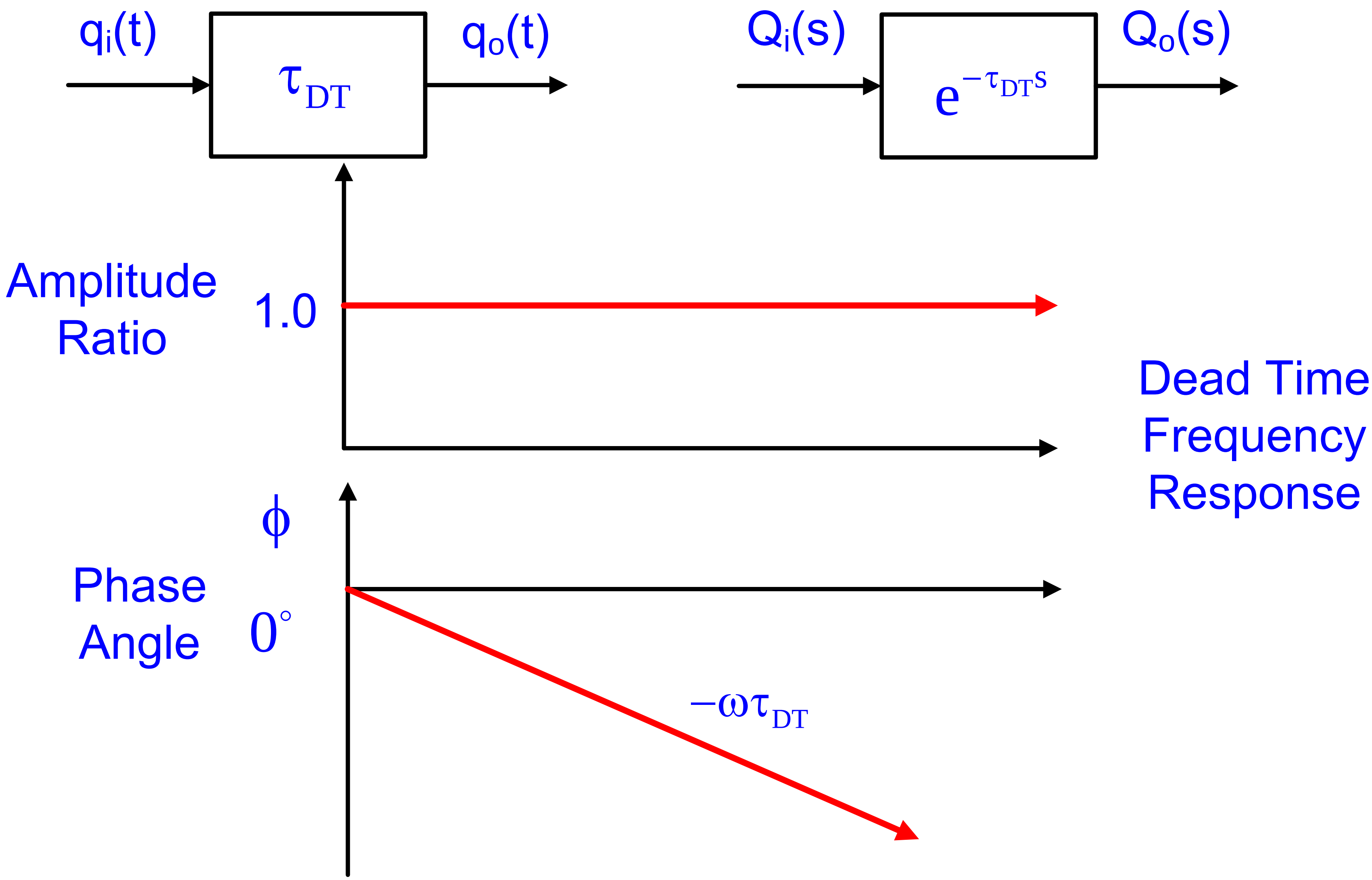
$$q_o(t) = \text{ output of dead-time element}$$

$$= q_i\left(t-\tau_{dt}\right)u\left(t-\tau_{dt}\right)$$

$$u\left(t-\tau_{dt}\right)=1 \quad \text{for } t \geq \tau_{dt}$$

$$u\left(t-\tau_{dt}\right)=0 \quad \text{for } t < \tau_{dt}$$

$$L\left[f\left(t-a\right)u\left(t-a\right)\right]=e^{-as}F\left(s\right)$$



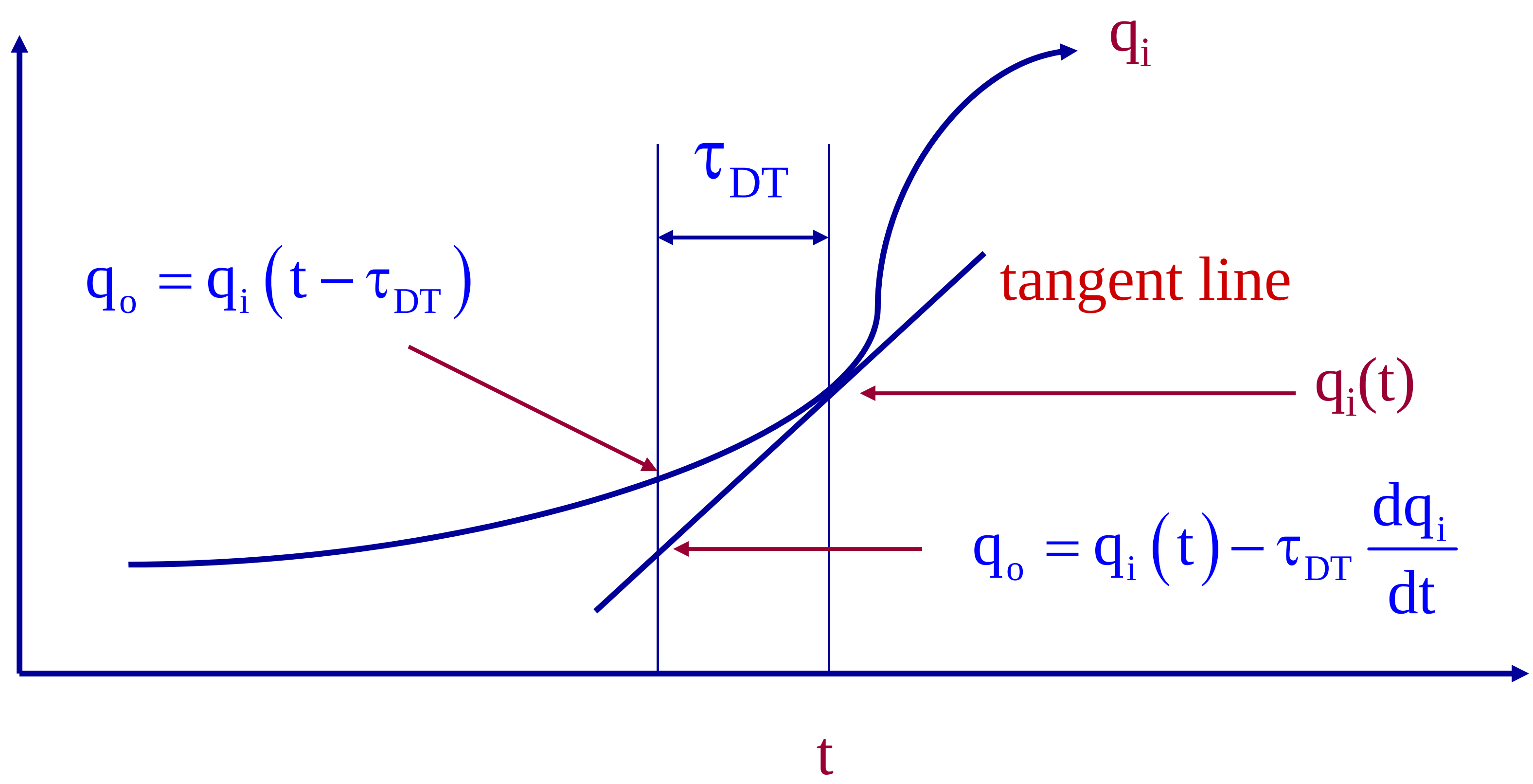
$q_i(t)$
τ_{DT}
$q_o(t)$
$Q_i(s)$
$e^{-\tau_{DT}s}$
$Q_o(s)$
Amplitude Ratio
1.0
Dead Time Frequency Response
ϕ
Phase Angle
0°
$-\omega\tau_{DT}$

- Dead-Time Approximations
  - The simplest dead-time approximation can be obtained graphically or by taking the first two terms of the Taylor series expansion of the Laplace transfer function of a dead-time element, τ_{DT}.

$$\frac{Q_o}{Q_i}(s) = e^{-\tau_{DT}s} \approx 1 - \tau_{DT}s \qquad q_o(t) \approx q_i(t) - \tau_{DT}\frac{dq_i}{dt}$$

  - The accuracy of this approximation depends on the dead time being sufficiently small relative to the rate of change of the slope of $q_i(t)$. If $q_i(t)$ were a ramp (constant slope), the approximation would be perfect for any value of τ_{DT}. When the slope of $q_i(t)$ varies rapidly, only small τ_{DT}'s will give a good approximation.

Dead-Time Graphical Approximation



- A frequency-response viewpoint gives a more general accuracy criterion; if the amplitude ratio and the phase of the approximation are sufficiently close to the exact frequency response curves of $e^{-\tau_{dt} s}$ for the range of frequencies present in $q_i(t)$, then the approximation is valid.
- The Pade' approximants provide a family of approximations of increasing accuracy (and complexity), the simplest two being:

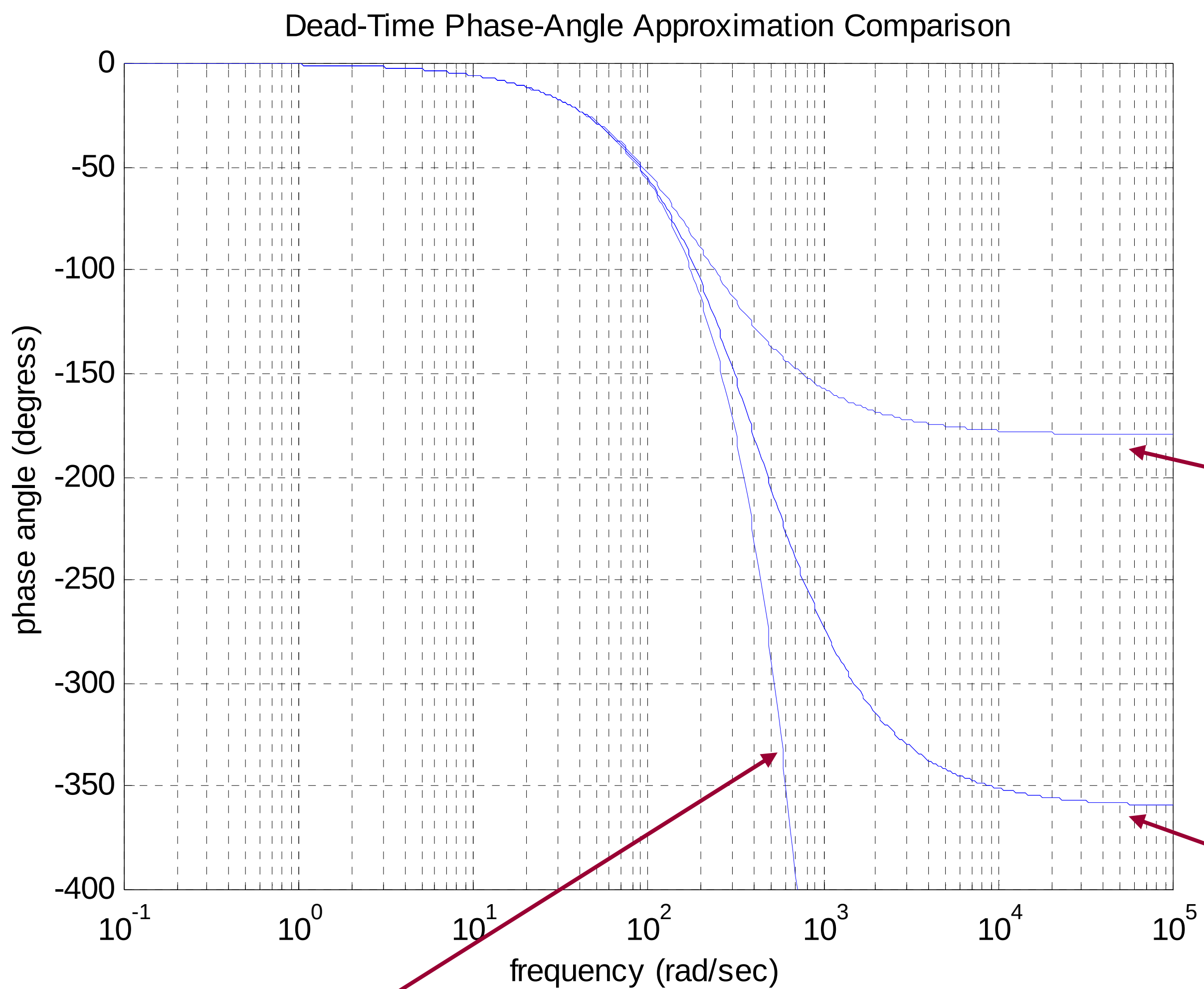
$$e^{-\tau s} = \frac{e^{\frac{-\tau s}{2}}}{e^{\frac{\tau s}{2}}} \approx \frac{1 - \frac{\tau s}{2} + \frac{\tau^2 s^2}{8} + \cdots + \frac{\left(-\frac{\tau s}{2}\right)^k}{k!}}{1 + \frac{\tau s}{2} + \frac{\tau^2 s^2}{8} + \cdots + \frac{\left(\frac{\tau s}{2}\right)^k}{k!}}$$

$$\frac{Q_o}{Q_i}(s) = \frac{2 - \tau_{dt} s}{2 + \tau_{dt} s}$$

$$\frac{Q_o}{Q_i}(s) = \frac{2 - \tau_{dt} s + \frac{(\tau_{dt} s)^2}{8}}{2 + \tau_{dt} s + \frac{(\tau_{dt} s)^2}{8}}$$

– Dead-time approximation comparison:

$$\tau_{dt} = 0.01$$



$$\frac{Q_o}{Q_i}(s) = \frac{2 - \tau_{dt}s}{2 + \tau_{dt}s}$$

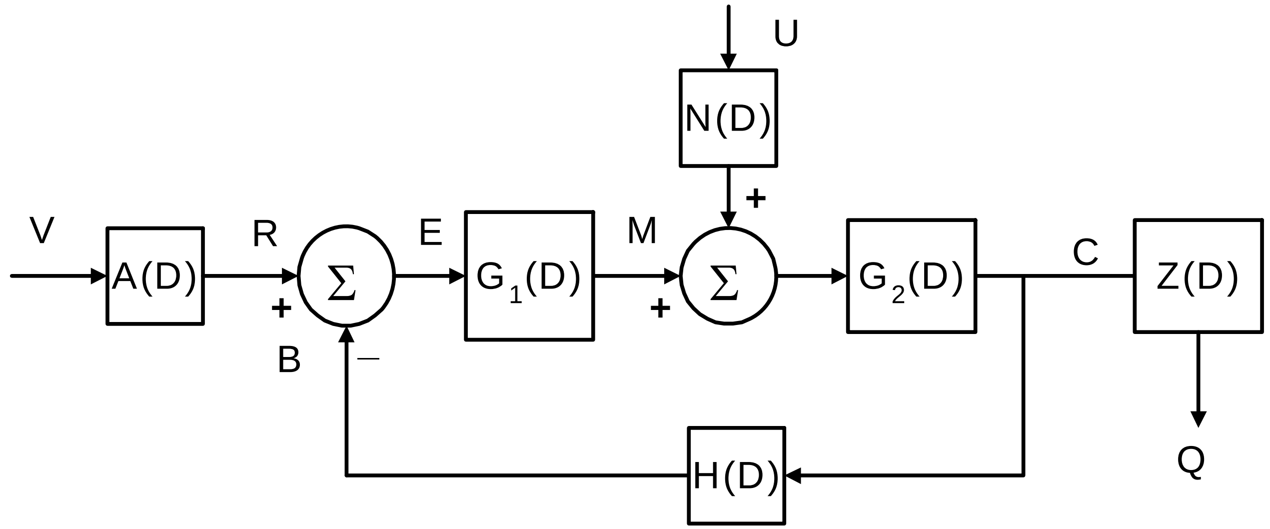
$$\frac{Q_o}{Q_i}(s) = \frac{2 - \tau_{dt}s + \frac{(\tau_{dt}s)^2}{8}}{2 + \tau_{dt}s + \frac{(\tau_{dt}s)^2}{8}}$$

$$e^{-\tau_{dt}s} = 1\angle - \omega\tau_{dt}$$

- Pade Approximation:
  - Transfer function is all pass, i.e., the magnitude of the transfer function is 1 for all frequencies.
  - Transfer function is non-minimum phase, i.e., it has zeros in the right-half plane.
  - As the order of the approximation is increased, it approximates the low-frequency phase characteristic with increasing accuracy.
- Another approximation with the same properties:

$$e^{-\tau s} = \frac{e^{\frac{-\tau s}{2}}}{e^{\frac{\tau s}{2}}} \approx \frac{\left(1 - \frac{\tau s}{2k}\right)^k}{\left(1 + \frac{\tau s}{2k}\right)^k}$$

Generalized Block Diagram



$$\left\{\left[A(D)V - \frac{H}{Z}(D)Q\right]G_1(D) + N(D)U\right\}G_2(D) = \frac{1}{Z(D)}Q$$

$$\left[1 + G_1G_2H(D)\right]Q = \left[AG_1G_2Z(D)\right]V + \left[NZG_2(D)\right]U$$

$$AG_1G_2Z(D) \equiv \frac{G_{nV}(D)}{G_{dV}(D)} \quad NZG_2(D) \equiv \frac{G_{nU}(D)}{G_{dU}(D)} \quad G_1G_2H(D) \equiv \frac{G_n(D)}{G_d(D)}$$

$$\left[1+G_1G_2H(D)\right]Q=\left[AG_1G_2Z(D)\right]V+\left[NZG_2(D)\right]U$$

$$\left(1+\frac{G_n}{G_d}\right)Q=\frac{G_{nV}}{G_{dV}}V+\frac{G_{nU}}{G_{dU}}U$$

$$\left(\frac{G_d+G_n}{G_d}\right)Q=\frac{G_{nV}}{G_{dV}}V+\frac{G_{nU}}{G_{dU}}U$$

$$Q=\frac{G_dG_{nV}}{(G_d+G_n)G_{dV}}V+\frac{G_{nU}G_d}{(G_d+G_n)G_{dU}}U$$

$$(G_d+G_n)G_{dV}G_{dU}Q=(G_dG_{nV}G_{dU})V+(G_{nU}G_dG_{dV})U$$

<u>Closed-Loop System Characteristic Equation</u>:

$$(G_d+G_n)G_{dV}G_{dU}=0$$

- The terms G_{dV} and G_{dU} are almost always themselves stable (no right-half plane roots) and when they are not stable it is generally obvious since these terms usually are already in factored form where unstable roots are apparent.
- For these reasons it is conventional to concentrate on the term $G_n + G_d$ which came from the original $1 + G_1G_2H$ term which describes the behavior of the feedback loop without including outside effects such as A(D), N(D), and Z(D).
- When we proceed in this fashion we are really examining the stability behavior of the closed loop rather then the entire system. Since instabilities in the outside the loop elements are so rare and also usually obvious, this common procedure is reasonable.

- We may write the system characteristic equation in a more general form:

$$1 + G_1 G_2 H(s) = 0$$

$$1 + \frac{G_n(s)}{G_d(s)} = 0$$

$$G_d(s) + G_n(s) = 0$$

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

- Assume that a_0 is nonzero, otherwise the characteristic equation has one or more zero roots which we easily detect and which do not correspond to stable systems.

- Routh Criterion Steps
  - Arrange the coefficients of the characteristic polynomial into the following array:

$$\begin{matrix} a_n & a_{n-2} & a_{n-4} & a_{n-6} \\ a_{n-1} & a_{n-3} & a_{n-5} & \cdots \end{matrix}$$

  - Then form a third row: $b_1 \quad b_2 \quad b_3 \quad \cdots$
  - Where

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \qquad b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_3 = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}} \qquad \cdots$$

- When the 3rd row has been completed, a 4th row is formed from the 2nd and 3rd in exactly the same fashion as the 3rd was formed from the 1st and 2nd. This is continued until no more rows and columns can be formed, giving a triangular sort of array.
- If the numbers become cumbersome, their size may be reduced by multiplying any row by any positive number.
- If one of the *a's* is zero, it is entered as a zero in the array. Although it is necessary to form the entire array, its evaluation depends always on only the 1st column.
- Routh's Criterion states that the number of roots not in the LHP is equal to the number of changes of algebraic sign in the 1st column.

- Thus a stable system must exhibit no sign change in first column.
- The Routh criterion does not distinguish between real and complex roots, nor does it give the specific numerical values of the unstable roots.
- Although the complete Routh procedure gives a correct result in every case, two special situations are worth memorizing as shortcuts:
  - If the original system characteristic equation itself shows any sign changes, there is really no point in carrying out the Routh procedure; the system will always be unstable.
  - If there are any gaps (zero coefficients) in the characteristic equation, the system is always unstable.

- Note, however, that a lack of gaps or sign changes is a necessary but not a sufficient condition for stability.

– Although not of much practical significance, since they rarely occur in practical problems, two special cases can occur mathematically:

  a) a term in the first column is zero but the remaining terms in its row are not all zero, causing a division by zero when forming the next row.

  b) all terms in the second or any further row are zero, giving the indeterminate from 0/0. This indicates pairs of equal roots with opposite signs located either on the real axis or on the imaginary axis.

– The solution for these two special cases is as follows:

- For case (a) substitute 1/x for *s* in the characteristic equation, then multiply by x^n, and form a new array. This method doesn't work when the coefficients of the original characteristic equation and the newly formed characteristic equation are identical. Another solution is to replace the 0 by a very small positive number ε, complete the array and then evaluate the signs in the first column by letting $\varepsilon \to 0$. Or another solution is to multiply the original polynomial by (s+1), which introduces an additional negative root, and then form the Routh array.

- For case (b) form an auxiliary equation using coefficients from the row above, being careful to alternate powers of *s*. Differentiate the equation with respect to s to obtain the coefficients of the previously all-zero row. The roots of the auxiliary equation are also roots of the characteristic equation. These roots occur in pairs. They may be imaginary (complex conjugates) or real and equal in magnitude, with one positive and one negative.

– Thus for a system to be stable, there must be no sign changes in the first column (to ensure that there are no roots in the RHP) and no rows of zeros (to ensure that there are no pairs of roots on the imaginary axis).

- For example, one sign change in the first column and a row of zeros would imply one real root in the RHP and one real root of the same magnitude in the LHP.
- In addition to answering yes-no questions concerning absolute stability, the Routh criterion is often useful in developing design guidelines helpful in making trade-off choices among system physical parameters.

Nyquist Stability Criterion

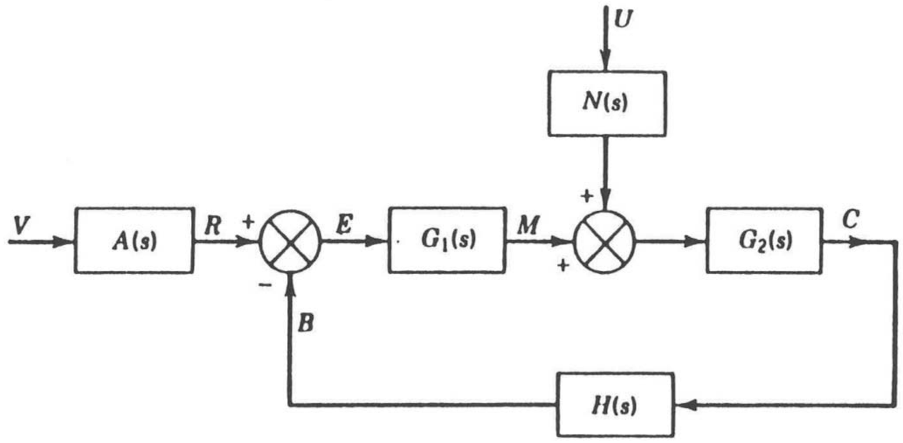
- The advantages of the Nyquist stability criterion over the Routh criterion are:
  - It uses the open-loop transfer function, i.e., (B/E)(s), to determine the number, not the numerical values, of the unstable roots of the closed-loop system characteristic equation. The Routh criterion requires the closed-loop system characteristic equation to determine the same information.

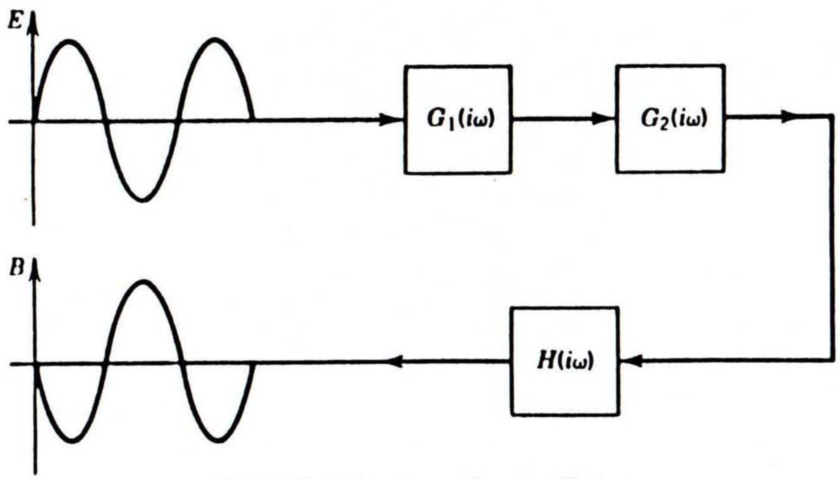
- If some components are modeled experimentally using frequency response measurements, these measurements can be used directly in the Nyquist criterion. The Routh criterion would first require the fitting of some analytical transfer function to the experimental data. This involves extra work and reduces accuracy since curve fitting procedures are never accurate.
- Being a frequency response method, the Nyquist criterion handles dead times without approximation since the frequency response of a dead time element, τ_{dt}, is exactly known, i.e., the Laplace transfer function of a dead time element is $e^{-\tau_{dt}s}$, with an amplitude ratio = 1.0 and a phase angle = - $\omega\tau_{dt}$.

- In addition to answering the question of absolute stability, Nyquist also gives some useful results on relative stability, i.e., gain margin and phase margin. Furthermore, the graphical plot used, keeps the effects of individual pieces of hardware more apparent (Routh tends to "scramble them up") making needed design changes more obvious.

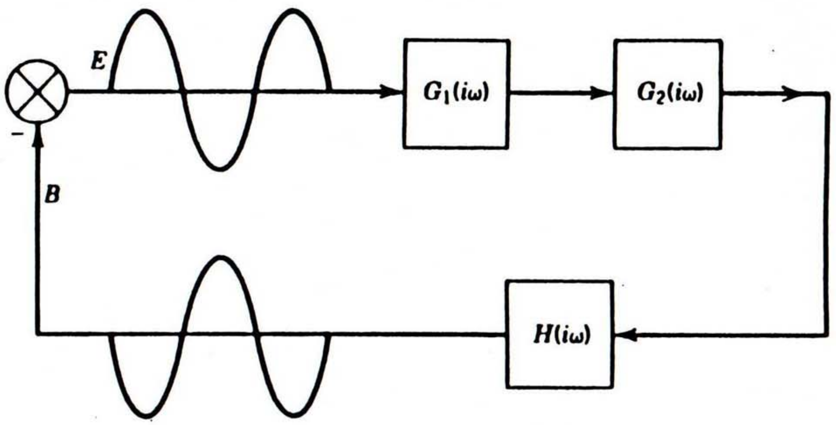
- While a mathematical proof of the Nyquist stability criterion is available, here we focus on its application and first give a simple explanation of its plausibility.

Feedback Control System Block Diagram





Externally driven open-loop oscillation



Self-excited closed-loop oscillation

Plausibility Demonstration for the Nyquist Stability Criterion

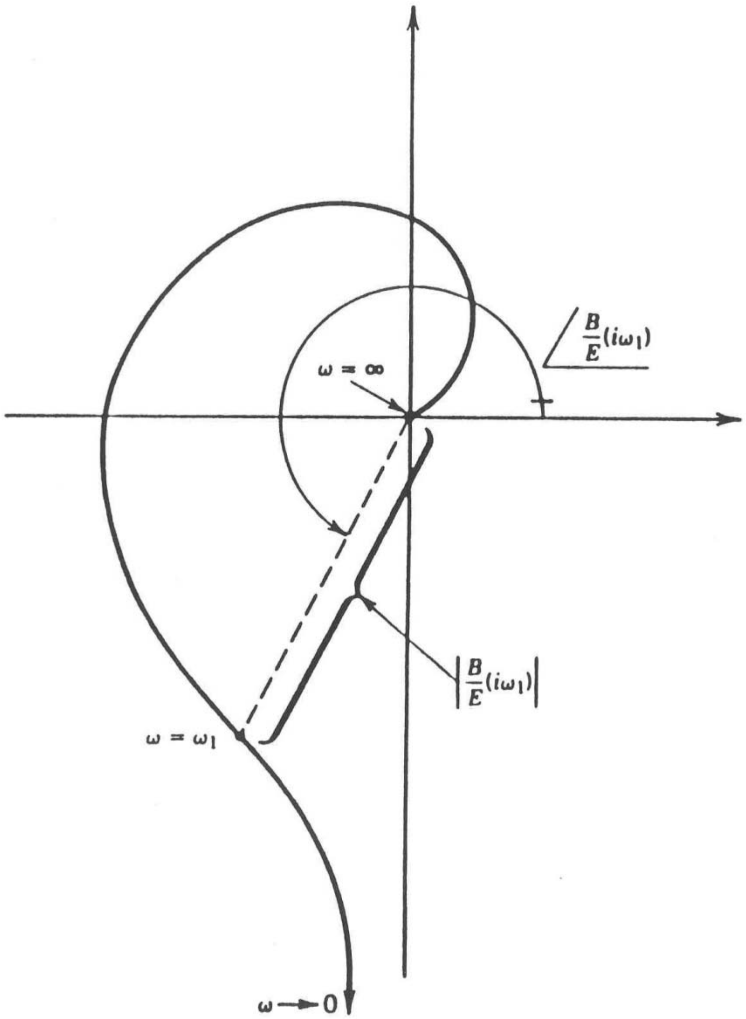
- Consider a sinusoidal input to the open-loop configuration. Suppose that at some frequency, $(B/E)(i\omega) = -1 = 1 \angle 180^\circ$. If we would then close the loop, the signal -B would now be exactly the same as the original excitation sine wave E and an external source for E would no longer be required. The closed-loop system would maintain a steady self-excited oscillation of fixed amplitude, i.e., marginal stability.
- It thus appears that if the open-loop curve $(B/E)(i\omega)$ for any system passes through the -1 point, then the closed-loop system will be marginally stable.

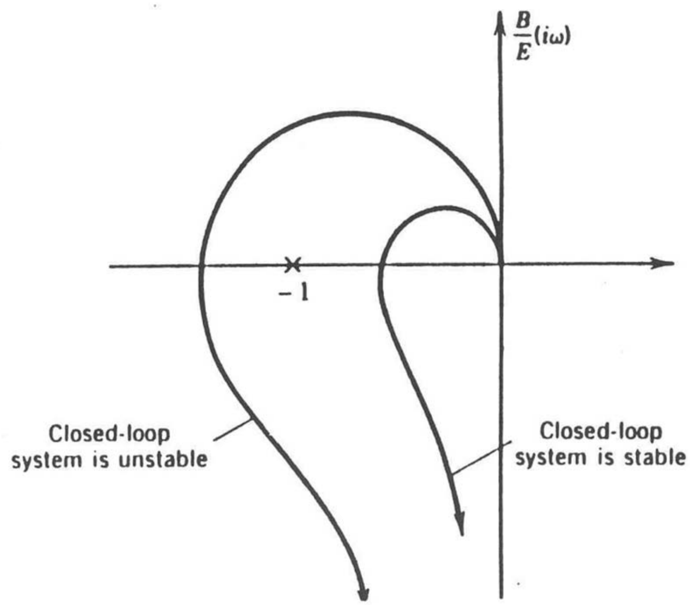
- However, the plausibility argument does not make clear what happens if curve does not go exactly through -1. The complete answer requires a rigorous proof and results in a criterion that gives exactly the same type of answer as the Routh Criterion, i.e., the number of unstable closed-loop roots.
- Instead, we state a step-by-step procedure for the Nyquist criterion.

1. Make a polar plot of (B/E)(iω) for $0 \leq \omega < \infty$, either analytically or by experimental test for an existing system. Although negative ω's have no physical meaning, the mathematical criterion requires that we plot (B/E)(-iω) on the same graph. Fortunately this is easy since (B/E)(-iω) is just a reflection about the real (horizontal) axis of (B/E)(+iω).

Polar Plot of Open-Loop Frequency Response

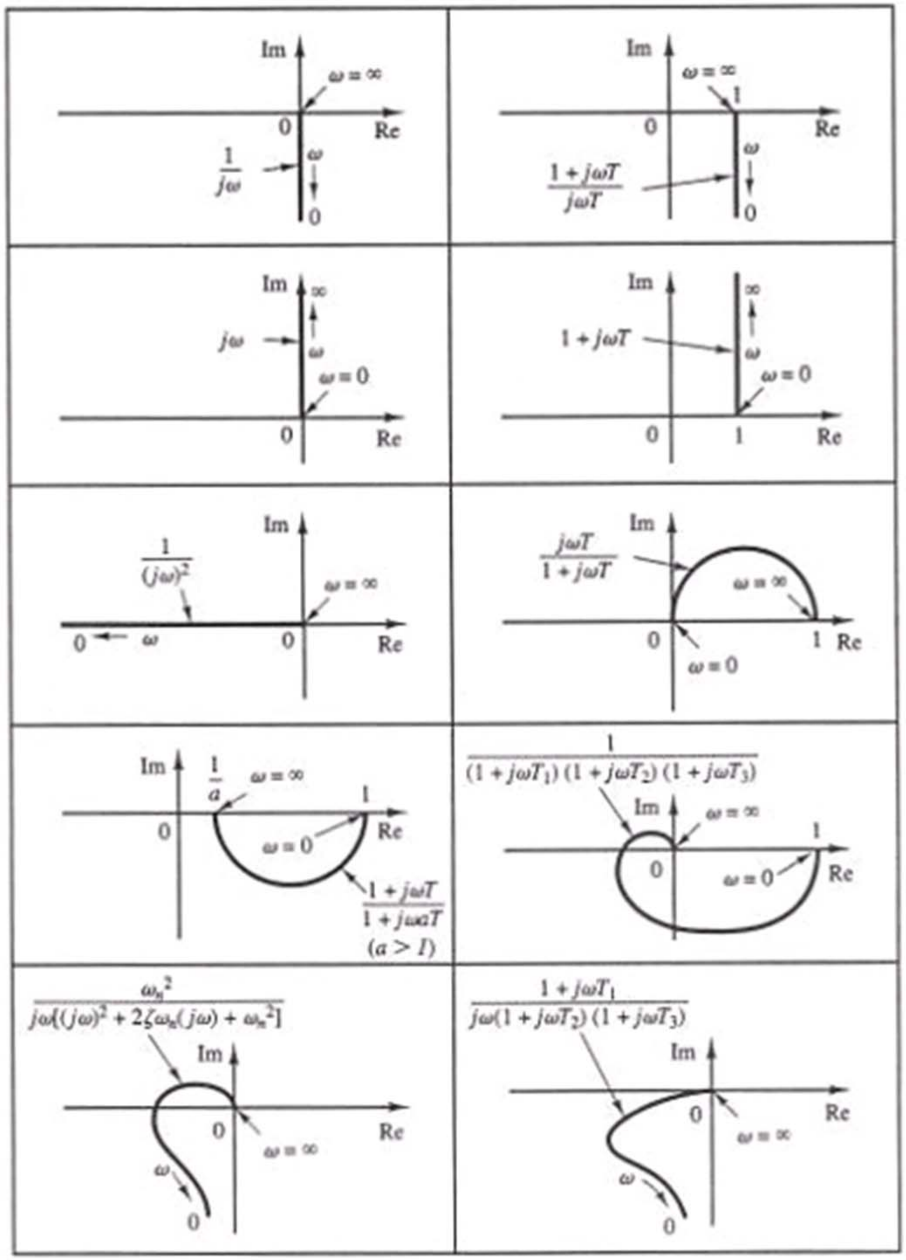
$$\frac{B}{E}(i\omega) = G_1G_2H(i\omega)$$



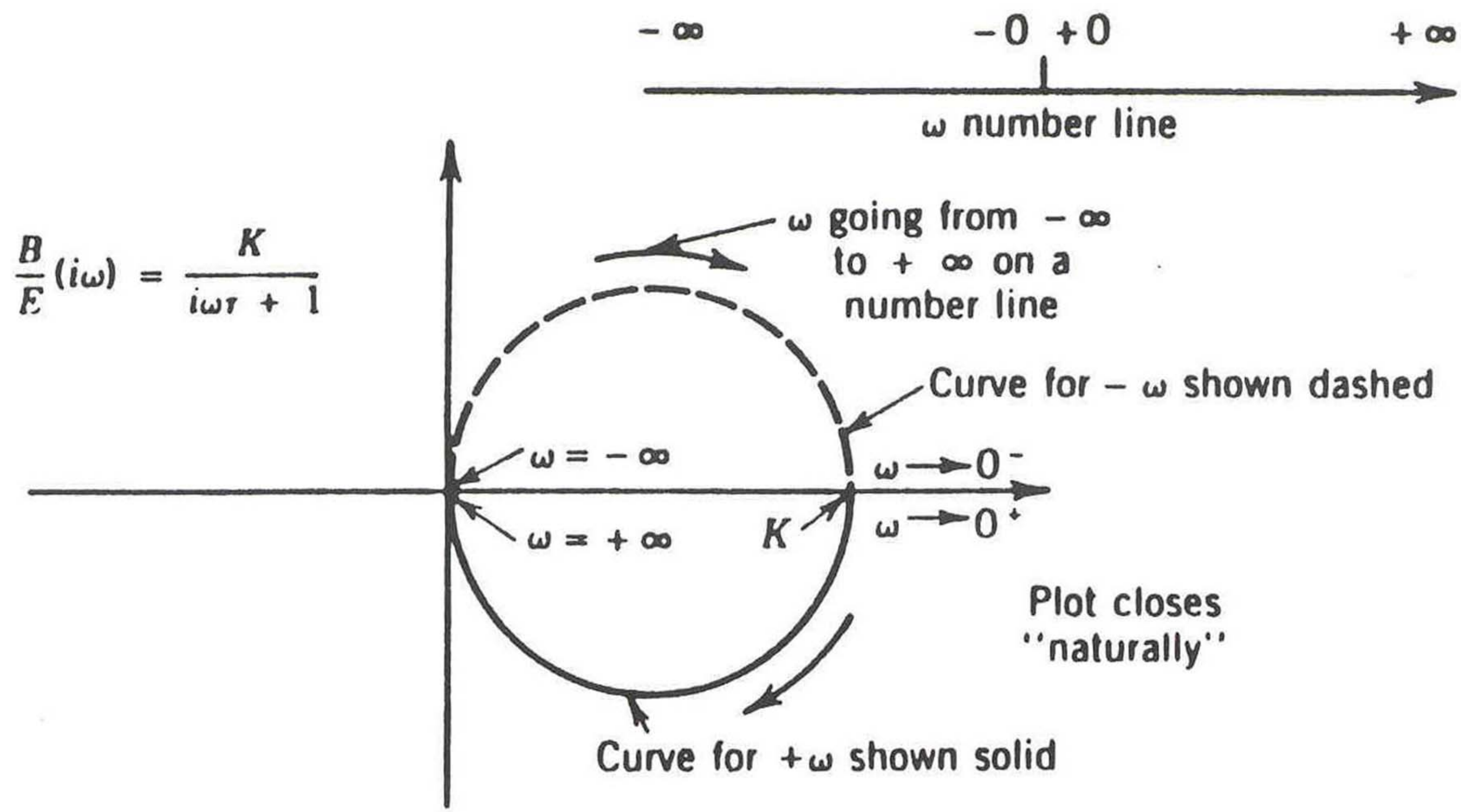


Simplified Version of Nyquist Stability Criterion

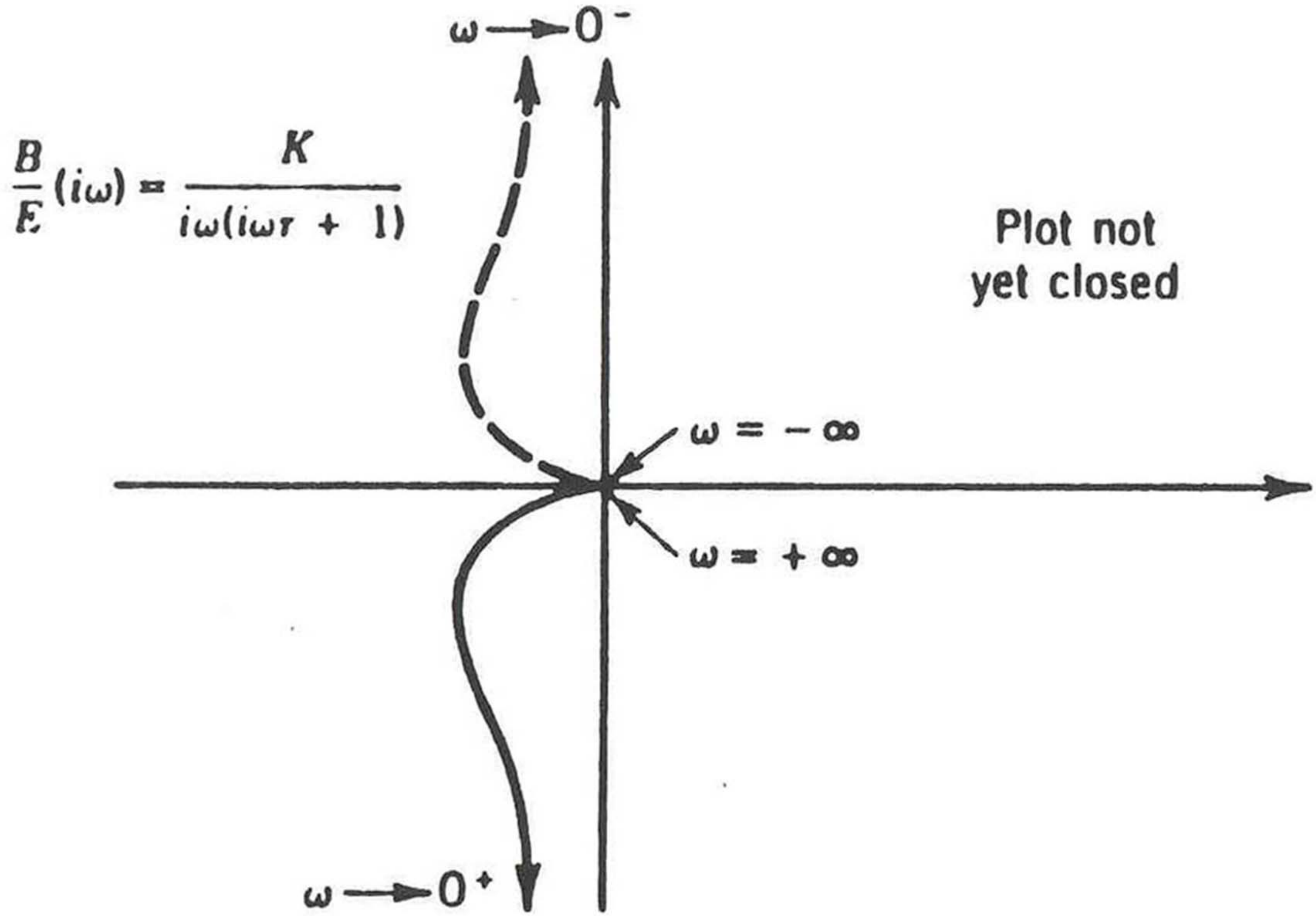
Examples of Polar Plots

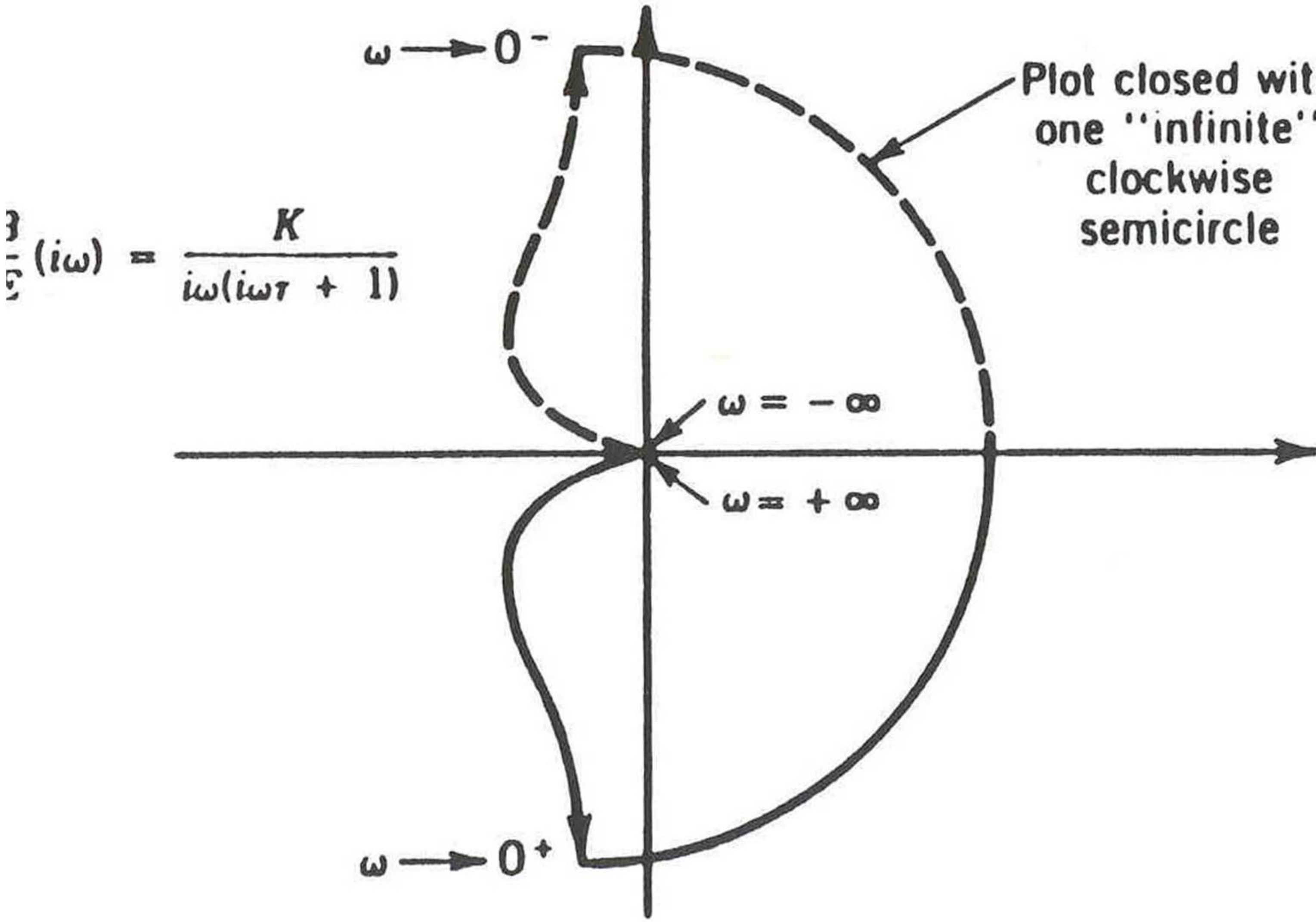


2. If (B/E)(iω) has no terms $(i\omega)^k$, i.e., integrators, as multiplying factors in its denominator, the plot of (B/E)(iω) for $-\infty < \omega < \infty$ results in a closed curve. If (B/E)(iω) has $(i\omega)^k$ as a multiplying factor in its denominator, the plots for $+\omega$ and $-\omega$ will go off the paper as $\omega \to 0$ and we will not get a single closed curve. The rule for closing such plots says to connect the "tail" of the curve at $\omega \to 0^-$ to the tail at $\omega \to 0^+$ by drawing k clockwise semicircles of "infinite" radius. Application of this rule will always result in a single closed curve so that one can start at the $\omega = -\infty$ point and trace completely around the curve toward $\omega = 0^-$ and $\omega = 0^+$ and finally to $\omega = +\infty$, which will always be the same point (the origin) at which we started with $\omega = -\infty$.



$-\infty$
$-0 \; +0$
$+\infty$
ω number line
$\frac{B}{E}(i\omega) = \frac{K}{i\omega\tau + 1}$
ω going from $-\infty$ to $+\infty$ on a number line
Curve for $-\omega$ shown dashed
$\omega = -\infty$
$\omega \to 0^-$
$\omega = +\infty$
K
$\omega \to 0^+$
Plot closes "naturally"
Curve for $+\omega$ shown solid

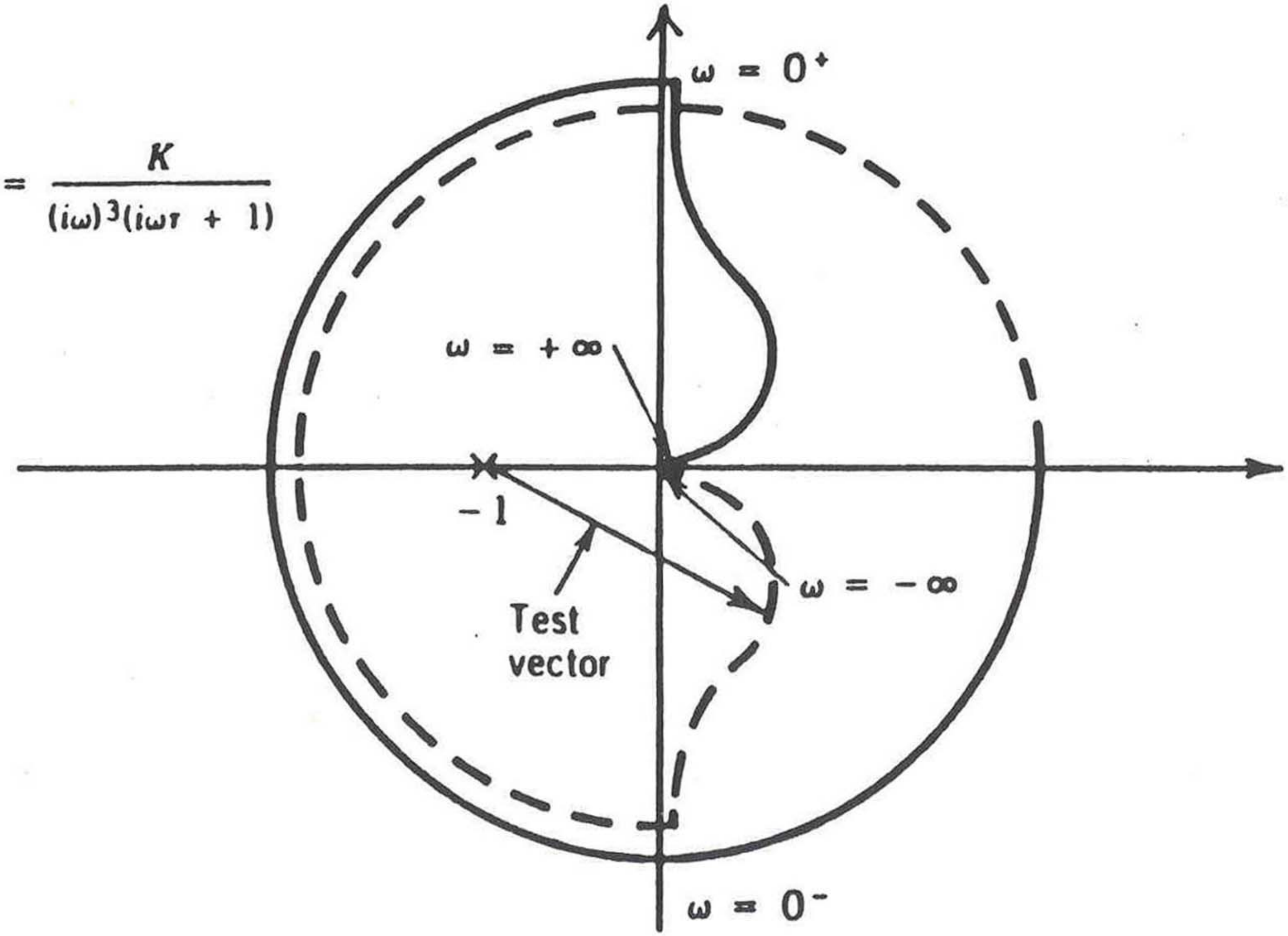

$\omega \rightarrow 0^-$
$\frac{B}{E}(i\omega) = \frac{K}{i\omega(i\omega\tau + 1)}$
Plot not yet closed
$\omega = -\infty$
$\omega = +\infty$
$\omega \rightarrow 0^+$


ω → 0⁻
Plot closed with one ''infinite'' clockwise semicircle
(iω) = K / iω(iωτ + 1)
ω = −∞
ω = +∞
ω → 0⁺

3. We must next find the number N_p of poles of $G_1G_2H(s)$ that are in the right half of the complex plane. This will almost always be zero since these poles are the roots of the characteristic equation of the open-loop system and open-loop systems are rarely unstable. If the open-loop poles are not already factored and thus apparent, one can apply the Routh criterion to find out how many unstable ones there are, if any. If $G_1G_2H(i\omega)$ is not known analytically but rather by experimental measurements on an existing open-loop system, then it must have zero unstable roots or else we would never have been able to run the necessary experiments because the system would have been unstable. We thus generally have little trouble finding N_p and it is usually zero.

4. We now return to our plot (B/E)(iω), which has already been reflected and closed in earlier steps. Draw a vector whose tail is bound to the -1 point and whose head lies at the origin, where $\omega = -\infty$. Now let the head of this vector trace completely around the closed curve in the direction from w = $-\infty$ to 0- to 0+ to $+\infty$, returning to the starting point. Keep careful track of the total number of net rotations of this test vector about the -1 point, calling this N_{p-z} and making it positive for counter-clockwise rotations and negative for clockwise rotations.

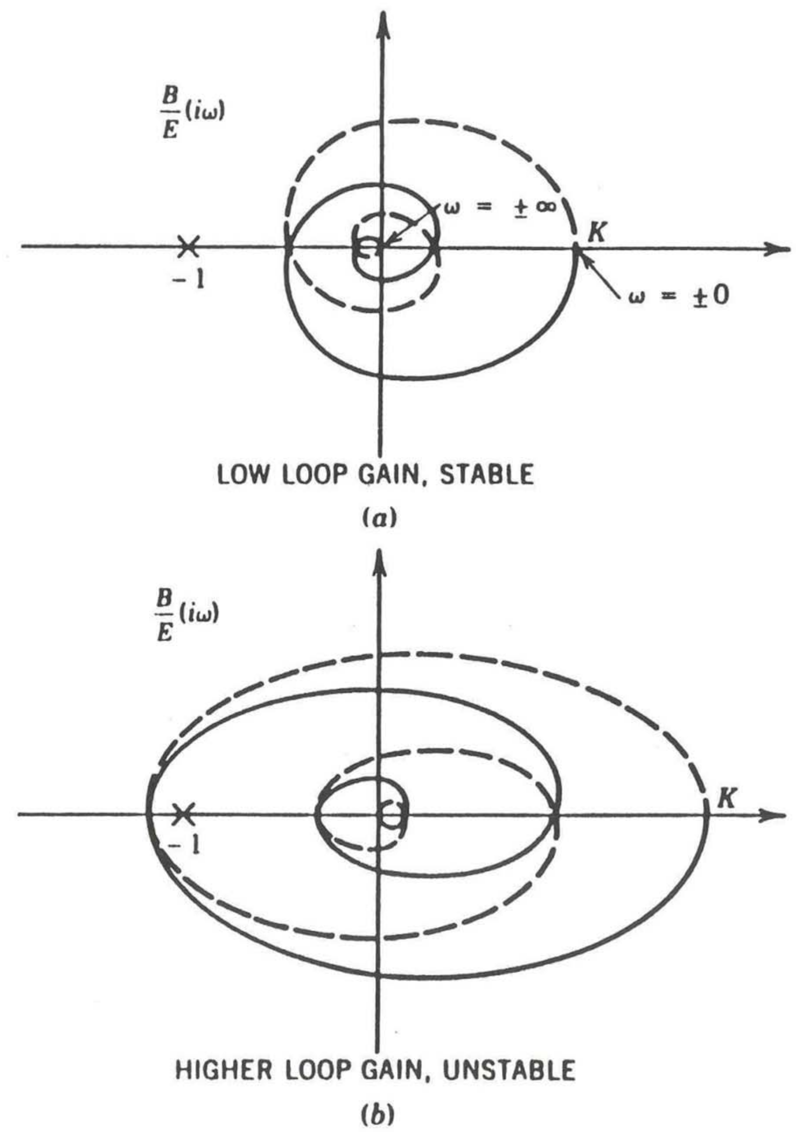
$$\frac{B}{E}(i\omega) = \frac{K}{(i\omega)^3(i\omega\tau + 1)}$$



5. In this final step we subtract N_{p-z} from N_p. This number will always be zero or a positive integer and will be equal to the number of unstable roots for the closed-loop system, the same kind of information given by the Routh criterion. The example shows an unstable closed-loop system with two unstable roots since $N_p = 0$ and $N_{p-z} = -2$.

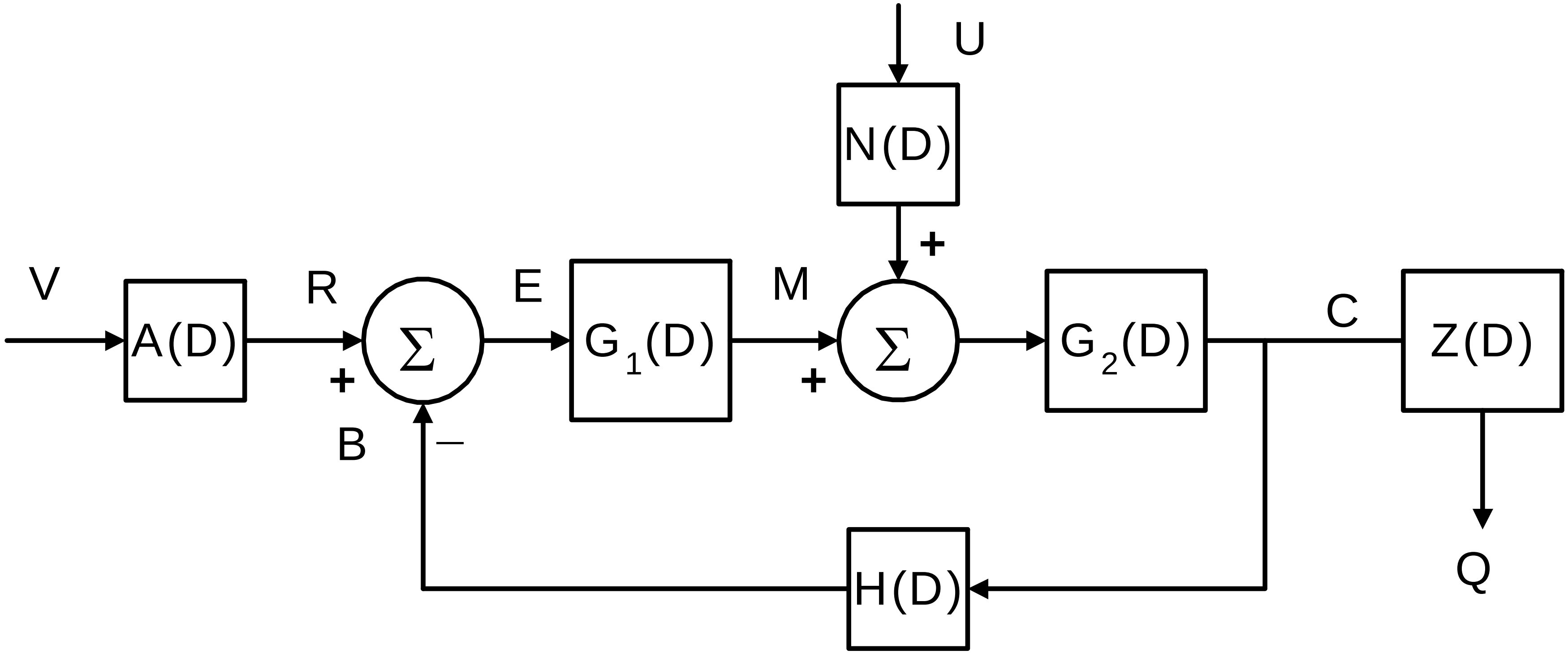
- The Nyquist criterion treats without approximation systems with dead times. Since the frequency response of a dead time element τ_{dt} is given by the expression $1\angle -\omega\tau_{dt}$, the (B/E)(iω) for the system of Figure (a) spirals unendingly into the origin. With low loop gain, the closed-loop system is stable, i.e., $N_p = 0$ and $N_{p-z} = 0$.
- Raising the gain, Figure (b) next page, expands the spirals sufficiently to cause the test vector to experience two net rotations, i.e., $N_{p-z} = -2$, causing closed-loop instability. Further gain increases expand more and more of these spirals out to the region beyond the -1 point, causing N_{p-z} to increase, indicating the presence of more and more unstable closed-loop roots.

Nyquist Stability Analysis of a System with Dead Time



Root-Locus Interpretation of Stability

- The root locus method for analysis and design is a method to find information about closed-loop behavior given the open-loop transfer function.
- The root locus is a plot of the poles of the closed-loop transfer function as any single parameter varies from 0 to ∞.
- The most straightforward method to obtain the root locus is simply to vary the parameter value and use a polynomial root solver to find the poles. However, early techniques in control analysis still give important insights into the design of closed-loop systems.



Characteristic Equation of the Closed-Loop System

$$1+KG_1(s)G_2(s)H(s)=0$$

K is the parameter that is being varied from 0 to ∞.

- The root locus begins at the poles of the open-loop transfer function $KG_1(s)G_2(s)H(s)$ and ends at the zeros of the open-loop transfer function or at infinity.
- Rewrite the closed-loop transfer function as

$$KG_1(s)G_2(s)H(s) = -1$$

- This implies that

$$\left|KG_1(s)G_2(s)H(s)\right| = 1$$

$$\angle G_1(s)G_2(s)H(s) = \pm(2k+1)\pi \qquad k = 0,\ 1,\ 2,\ \cdots$$

- For a point s^* in the s plane to be a part of the root locus, the total angle from the poles and zeros of $G_1(s)G_2(s)H(s)$ to s^* must be $\pm (2k+1)\pi$.
- The gain K that corresponds to this point is found by:

$$K = \frac{1}{\left| G_1\left(s^*\right) G_2\left(s^*\right) H\left(s^*\right) \right|}$$

- Consider as an example a system with open-loop transfer function:

$$\frac{B}{E}(s) = \frac{K}{s\left(\tau_1 s + 1\right)\left(\tau_2 s + 1\right)}$$

- The closed-loop characteristic equation is given by:

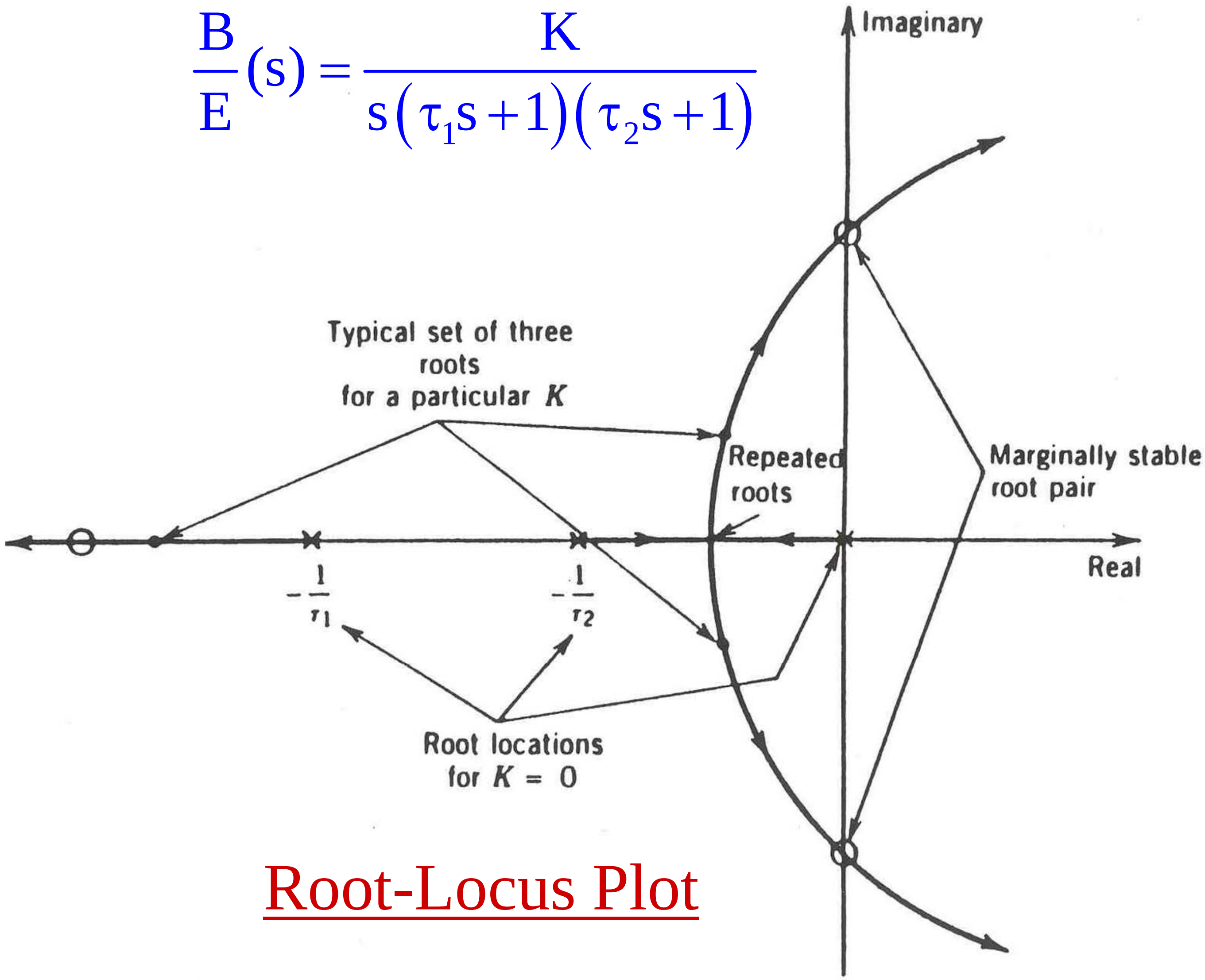
$$\tau_1\tau_2 s^3 + \left(\tau_1 + \tau_2\right)s^2 + s + K = 0$$

- Assume that τ_1 and τ_2 have been chosen and we wish to explore the effect of varying loop gain K on system stability. For each value of K, the equation has 3 roots which may be plotted in the complex plane. For K = 0, these roots are 0, $-1/\tau_1$, $-1/\tau_2$. As K is increased, the roots trace out continuous curves that are called the root loci.
- Every linear, time-invariant feedback system has a root-locus plot and these are extremely helpful in system design and analysis.

Root Locus method gives information about closed-loop behavior given the open-loop transfer function.

The root locus is a plot of the poles of the closed-loop transfer function as any single parameter varies from 0 to ∞.

$$\frac{B}{E}(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$



Root-Locus Plot